VIRTUAL COMPLETE INTERSECTIONS OF POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$

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INTRODUCTION

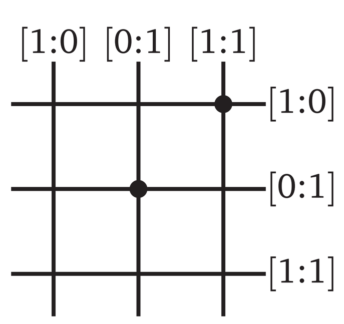
- We consider configurations of points in $\mathbb{P}^1 \times \mathbb{P}^1$
- While free resolutions are standard, virtual resolutions better convey geometry
- We look for configurations with the simplest virtual resolutions, a "Koszul Complex"

PRELIMINARIES

- The **biprojective space** $\mathbb{P}^1 \times \mathbb{P}^1$ is the set

$$\{((a_0,a_1),(b_0,b_1)) \in \mathbb{C}^2 \times \mathbb{C}^2|_{(a_0,a_1),(b_0,b_1)\neq(0,0)}\}/\sim$$

$x \sim y \iff x = \lambda y$, where $x, y \in \mathbb{P}^1, \lambda \in \mathbb{C}^*$.



- ⋆ Each ruling represents a copy of $\mathbb{P}^1$
- ⋆ $\mathbb{Z}^2$-Graded Cox ring: $\mathbb{C}[x_1, x_2, y_1, y_2]$
- ⋆ $\deg(x_i) = (1,0)$ and $\deg(y_i) = (0,1)$
- ⋆ Varieties: zero locus of homogeneous polynomials

- **Irrelevant ideal:** $B = \langle x_0, x_1\rangle \cap \langle y_0, y_1\rangle \longleftrightarrow V(B) = \emptyset$
- $\{V(I) \neq \emptyset\} \underset{V}{\overset{I}{\rightleftarrows}} \{$radical homogeneous B-saturated ideals $\subset S\}$
- A **virtual resolution** for an ideal I in $\mathbb{P}^1 \times \mathbb{P}^1$ is a free complex:

$$0 \longleftarrow I \xleftarrow{\varphi_0} S \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \xleftarrow{\varphi_3} \cdots$$

such that F_i are free modules for $i \geq 0$, $\operatorname{ann}\left(\frac{\ker(\varphi_i)}{\operatorname{im}(\varphi_{i+1})}\right) \supseteq B^l$, and $\operatorname{im}(\varphi_1) : B^\infty = I : B^\infty$.

- An ideal I of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is a **virtual complete intersection (VCI)** if I has a virtual resolution that is a Koszul complex.
- In particular, I will have a virtual resolution of the form

$$0 \longleftarrow I \longleftarrow S \longleftarrow S(-a,-b) \oplus S(-c,-d) \longleftarrow S(-a-c,-b-d) \longleftarrow 0$$

- Geometrically, $V(I) = V(f) \cap V(g)$, where f, g are homogeneous
- **Bigraded Bézout's Theorem**: If f and g are in general position of multidegree (a,b) and (c,d) respectively, then $|V(f) \cap V(g)| = ad + bc$ counting multiplicities.

QUESTION

Let X be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Determine whether X is a VCI based on combinatorial properties of X.

- Let m (resp. n) be the **maximum number of points** on a single horizontal (resp. vertical) ruling.
- We consider two sets of points **equivalent up to configuration** if they are the same under permutation and relabeling of the rulings.

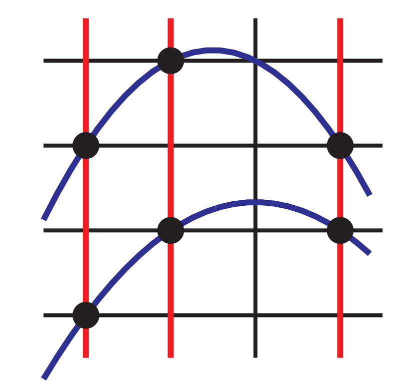
SUMMARY OF RESULTS

1. Same # of points on each ruling $\implies$ VCI
2. When $|X| < mn$, restrictions on what VCIs must look like
   - Cross Point Condition
   - Number Theoretic Results
3. Actual values of the coordinates can affect VCI, too.

SUFFICIENT CONDITION

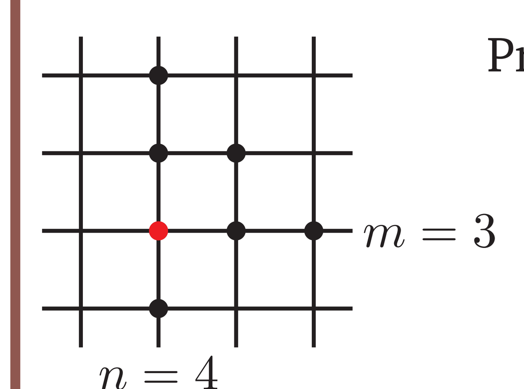
If X has the same number (n) of points in each vertical (or each horizontal) ruling, it is a VCI.

- k vertical rulings each having n points
  $\deg(f) = (n, \leq n)$,
  $\deg(g) = (0, k)$.
- Proof idea: Use Lagrangian interpolation



NECESSARY CONDITIONS WHEN $|X| < mn$

If $|X| < mn$, and $\exists$ a point in X on a horizontal ruling l_1 with m points and vertical ruling l_2 with n points, then X is not a VCI.



Proof idea:

1. If $X = V(f) \cap V(g)$, then it must be the case: $deg(f) = (\geq m, \geq n)$, and g contains l_1 and l_2;
2. The intersection of $V(f)$ and $V(g) \setminus \{l_1, l_2\}$ contradicts Bigraded Bézout's Theorem.

MORE NECESSARY CONDITIONS

Let X be a VCI with $|X| < mn$.

- f has degree (m,n) and g has vertical and horizontal components exactly on rulings with m and n points
- $\gcd(m,n)$ divides $|X|$
- If $\gcd(m,n) = 1$: g has degree:

$$(n^{-1}|X| \mod m, \quad m^{-1}|X| \mod n)$$

Proof idea: Chicken McNugget Theorem.

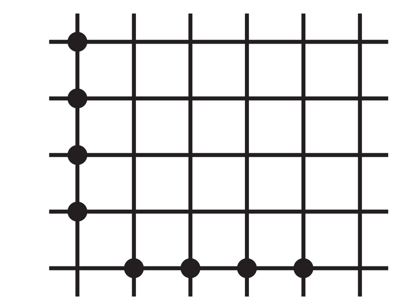
CLASSIFICATION FOR TWO RULING CASE

If all points lie on two horizontal (resp. vertical) rulings, they form a VCI if and only if one of the following holds:

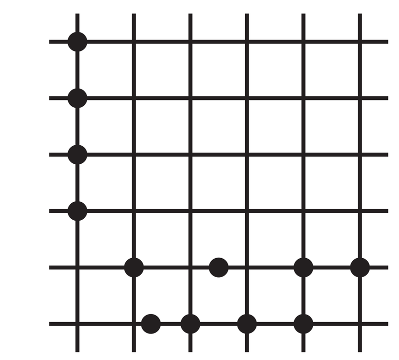
- No two of them lie on the same vertical (resp. horizontal) ruling
- Both horizontal (resp. vertical) rulings contain the same number points

RESULTS IN ACTION

If $|X| < mn, m = 4, n = 4$, the only VCI configurations are as shown:



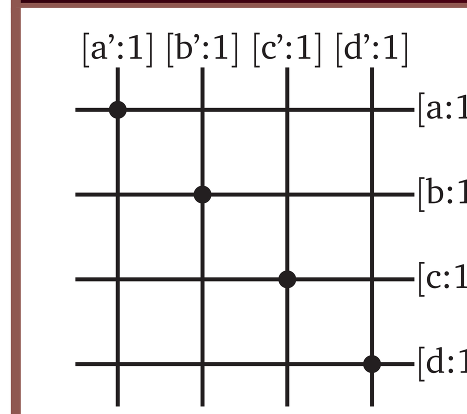
8-point VCI



12-point VCI

- By Cross Point Condition, m and n points share no coordinates
- By GCD condition, $|X|$ is 8 or 12
- f has degree $(4,4)$ and g contains vertical and horizontal form
- If $|X| = 12 = 4c + 4d$, rest of g must be $(1,0)$ or $(0,1)$ form
- Each such form must have 4 points of X

COORDINATE DEPENDENCE

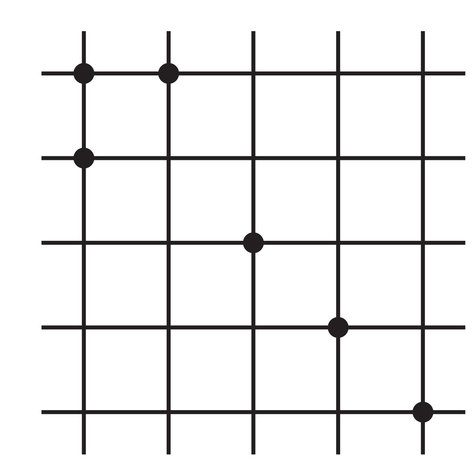


Configuration not sufficient to determine minimal free resolution; depends on whether four points have the same cross ratio in both copies of $\mathbb{P}^1$:

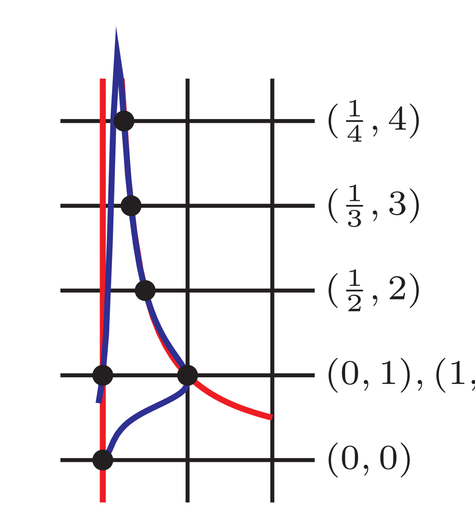
$$\frac{(c-a)(d-b)}{(d-a)(c-b)} = \frac{(c'-a')(d'-b')}{(d'-a')(c'-b')}$$

False Conjecture: Configuration is sufficient to determine VCIs.

Counterexample:



In general, not a VCI.



Red:(2,1); Blue:(2,2).

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