Toric Arrangements, Acyclic Orientations and the Chromatic Polynomial of Symmetric Graphs.

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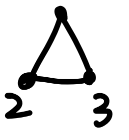
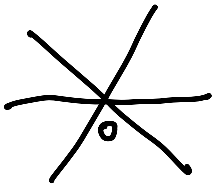
Type A Graphical Arr

Def: A simple graph G on n vertices $\{1, 2, \cdots, n\}$ has a graphical arr. in $\mathbb{R}^n$

$$\mathcal{A}(G) = \{x_i - x_j = 0 \mid i,j \in E(G)\}$$

Ex $G =$ 

$\mathcal{A}(G) =$ 

on $\{x_1 + x_2 + x_3 = 0\}$

6 Chambers

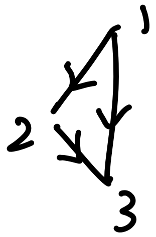
Note: $\mathcal{A}(G)$ essential $\iff$ G connected

\# Chambers in $\mathcal{A}(G)$? [Sta '73; Greene-Zas '83]

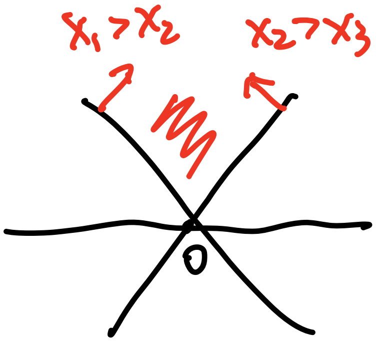
① **Thm:** There is a bijection b/w

$$\text{Chambers in } \mathcal{A}(G) \longleftrightarrow \text{Acyclic Orientations of } G$$

Ex:



$\leftrightsquigarrow$



② **Thm:** \# Chambers in $\mathcal{A}(G)$

$$= (-1)^n \chi_G(-1)$$

$\chi_G(q)$ chromatic polynomial

Ex: $G = \Delta$ $\qquad \chi_G(q) = q(q-1)(q-2)$

$$(-1)^3 \chi_G(-1) = \underline{\underline{6}}$$

Generalizations?

[Sta'73, G-Z'83]

[Develin-Macauley-Reiner '16, Novik-Pos-Sturmfels '02]

$$\begin{tikzcd}
\text{Type A} \arrow[r] \arrow[d] & \text{Toric Type A} \arrow[d] \\
\text{Type B} \arrow[r] & \text{Toric Type B}
\end{tikzcd}$$

[G-Z '83; Zas '82]

[Today's Talk]

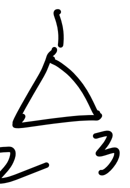
Type A Toric Graphical Arr.

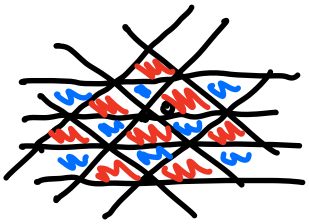
Def. Graph G as before

$$\mathcal{A}_{\text{aff}}(G) = \{ x_i - x_j \in \mathbb{Z} \mid ij \in E(G) \} \subseteq \mathbb{R}^n$$

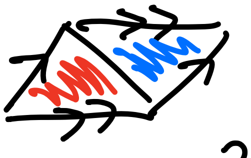
$$\mathcal{A}_{\text{tor}}(G) = \pi(\mathcal{A}_{\text{aff}}(G)) \subseteq \mathbb{R}^n/\mathbb{Z}^n$$

projection map $\pi: \mathbb{R}^n \longrightarrow \underbrace{\mathbb{R}^n/\mathbb{Z}^n}_{n\text{-torus}}$

Ex: $G=$ 

$\mathcal{A}_{\mathrm{aff}}(G)=$ 

on $\{x_1+x_2+x_3=0\}$

$\mathcal{A}_{\mathrm{tor}}(G)=$ 

2 chambers

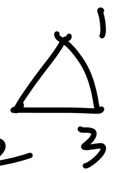
Chambers in $\mathcal{A}_{\mathrm{tor}}(G)$?

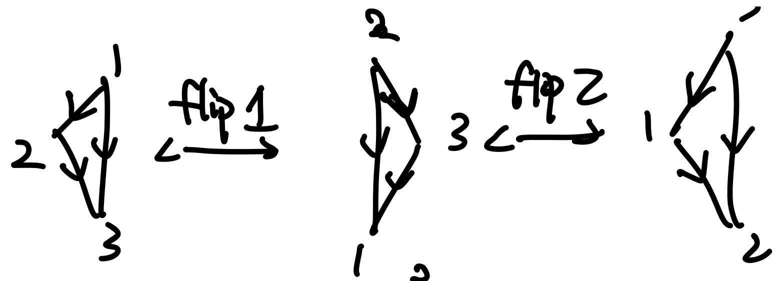
Equivalently, # Different Tiles in $\mathcal{A}_{\mathrm{aff}}(G)$?
(two tiles are the same if they only differ by a translation of an integer vector)

① Thm [DMR'16] There is a bijection b/w

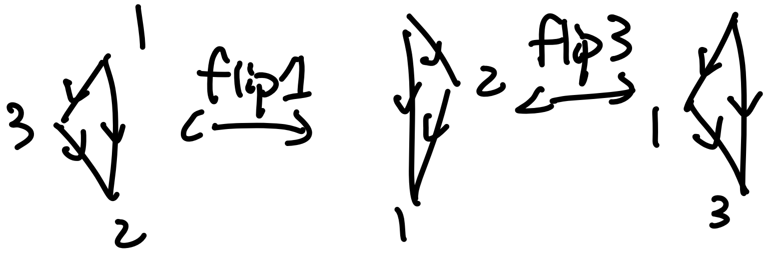
Chambers in $\mathcal{A}_{\mathrm{tor}}(G) \longleftrightarrow$ AOs of $G/\sim$

$\sim$: source-to-sink flips [Pretzel '86]
Reverse all arrows adj. to a source/sink

Ex: $G =$ 



2 equivalence classes



② Thm # Chambers in $\mathcal{A}_{tor}(G)$

$= |\text{linear coeff. of } \chi_G(q)|$ [Gebhard-Sagan '00]

$= T_G(1,0)$ → Tutte poly. [NPS '02]

Type B Graphical Arr.

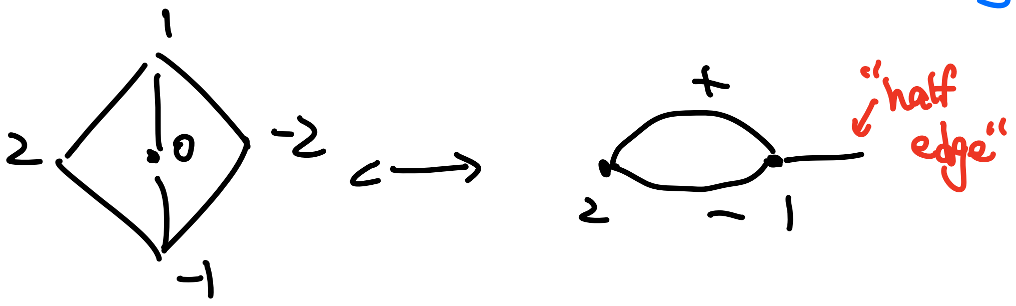
Need: Type B analogue of graphs

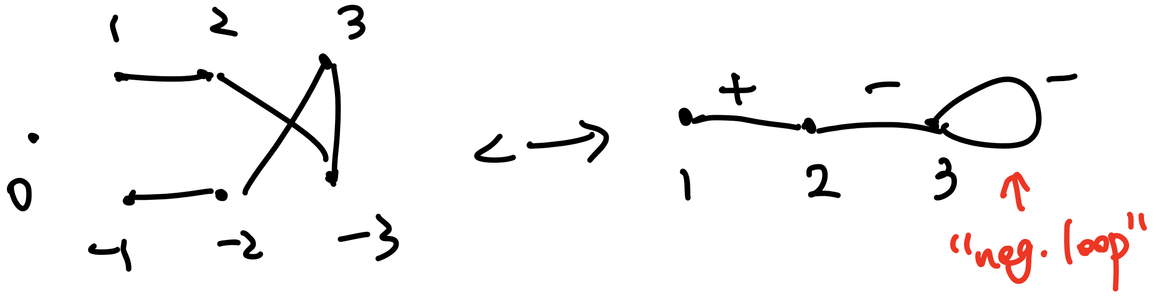
Def A symmetric graph G on $2n+1$ vertices

$\{-n, -n+1, \dots, -1, 0, 1, \dots, n\}$

is a simple graph s.t. $i,j \in E(G) \Leftrightarrow -i,-j \in E(G)$

Rmk: Sym. Graph $\Longleftrightarrow$ "Signed Graph" [Zas '83]



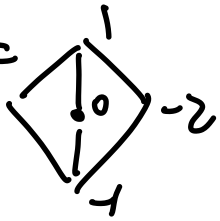


Def: Given Sym. Graph G

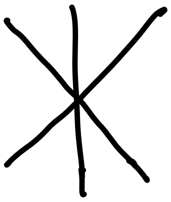
The Type B Graphical Arr. $\mathcal{B}(G)$ has hyperplanes as follows:

- $i, j \in E(G)$ $\quad x_i - x_j = 0$
- $i, -j \in E(G)$ $\quad x_i + x_j = 0$
- $i, 0 \in E(G)$ $\quad x_i = 0$
- $i, -i \in E(G)$ $\quad 2x_i = 0$

Ex. $G=$



$\mathcal{B}(G)=$



6 chambers

Lem: $\mathcal{B}(G)$ is essential $\Longleftrightarrow$ There is a path between every pair of vertices i & $-i$ in G

We say G is "weakly connected"

 # Chambers in $\mathcal{B}(G)$?

① **Def:** Given sym. graph G.

A sym. AO of G is an AO of G

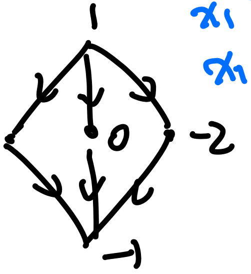
s.t. $i \to j \Longleftrightarrow -j \to -i$

Thm: There is a bij. b/w

Chambers in $\mathcal{B}(G) \longleftrightarrow$ Sym. AOs of G

Ex.

$G=$

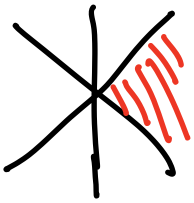


$x_1>0$

$x_1>x_2$

$x_1>-x_2$

$\longleftrightarrow$



② **Def:** A $\underline{\text{proper } k\text{-coloring}}$ of sym. graph G is a coloring $f: V(G) \longrightarrow \{0, \pm 1, \cdots, \pm k\}$

s.t. [Zas'82]

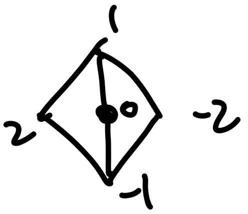
- $f(0) = 0$
- $f(i) = -f(-i)$
- $f(i) \neq f(j)$ if $i,j \in E(G)$

$$\# \text{ proper } k\text{-colorings} := \underbrace{\chi_G(k)}_{\text{chromatic poly.}}$$

Thm [Greene-Zas '83]

$$\# \text{ Chambers in } \mathcal{B}(G) = (-1)^n \chi_G(-1)$$

Ex: $G =$



$$\chi_G(k) = 2k(2k-1)$$

$$(-1)^2 \chi_G(-1) = \underline{\underline{6}}$$

Type B Toric Graphical Arr.

Def: Given Sym. Graph G.

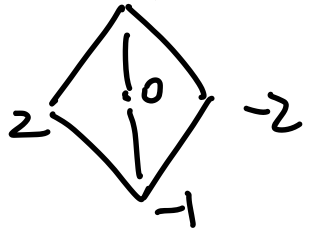
$\mathcal{B}_{aff}(G) \subseteq \mathbb{R}^n$ has hyperplanes

- $i, j \in E(G)$ $\quad x_i - x_j \in \mathbb{Z}$
- $i, -j \in E(G)$ $\quad x_i + x_j \in \mathbb{Z}$
- $i, 0 \in E(G)$ $\quad x_i \in \mathbb{Z}$
- $i, -i \in E(G)$ $\quad 2x_i \in \mathbb{Z}$

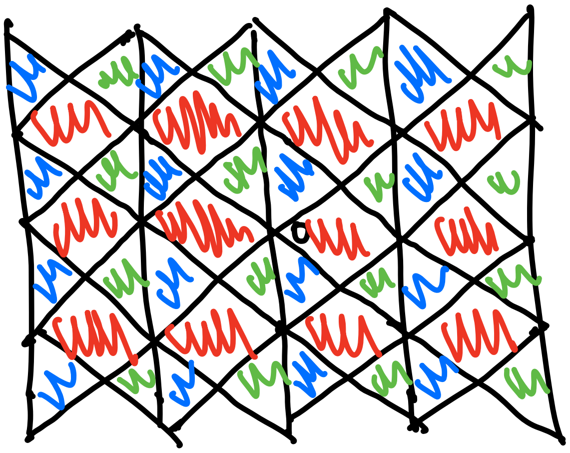
$\mathcal{B}_{tor}(G) = \pi(\mathcal{B}_{aff}(G)) \subseteq \mathbb{R}^n/\mathbb{Z}^n$

$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n$

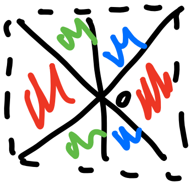
Ex: $G =$



$\mathcal{B}_{aff}(G) =$



$\mathcal{B}_{tor}(G) =$



3 <u>chambers</u>

Chambers in $\mathcal{B}_{tor}(G)$?

(# Diff. Tiles in $\mathcal{B}_{aff}(G)$?)

① <u>Thm</u>: There is a bij. b/w

Chambers in $\mathcal{B}_{tor}(G) \longleftrightarrow$ Sym. Aos of $G/\sim$

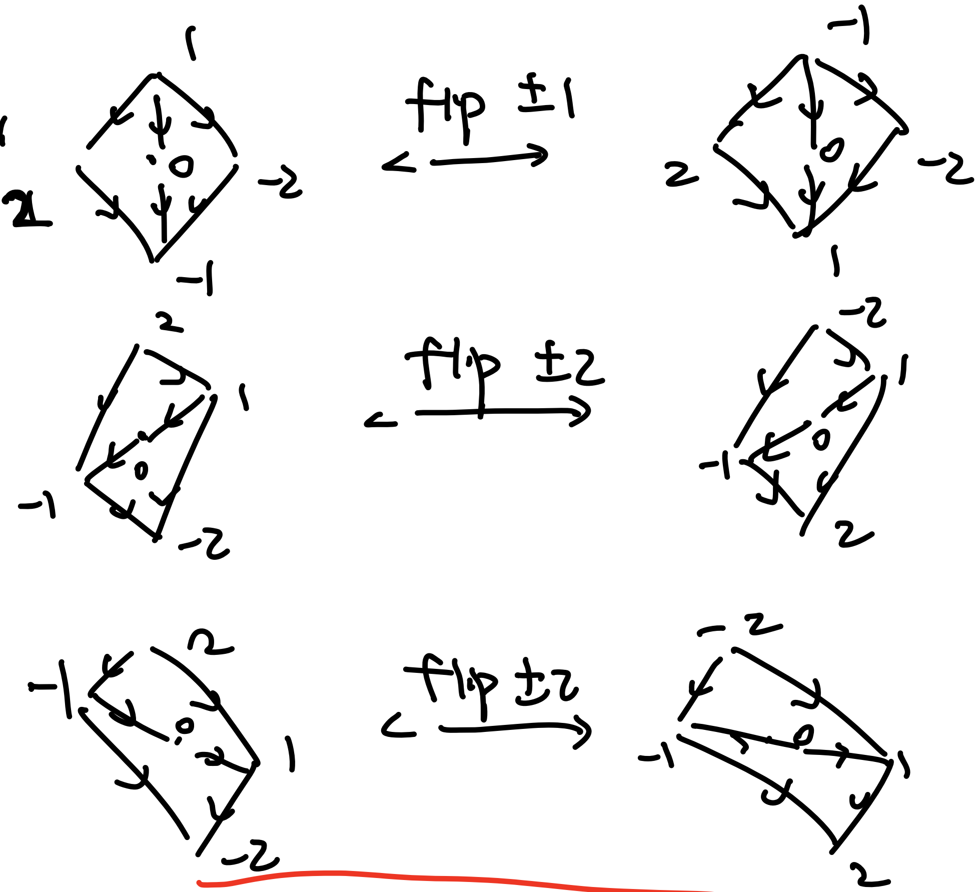
$\sim$: source-to-sink flips

If i is a source, then $-i$ is a sink

The flip reverses all arrows adj. to i or $-i$

<u>Note</u>: if $i, -i \in E(G)$, <u>NOT</u> allowed to flip

<u>Ex.</u>

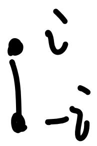
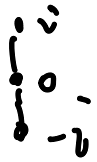


$\boxed{3 \text{ equiv. classes}}$

② Chromatic Polynomial?

- $\#$ Chan. in $\mathcal{B}_{tor}(G) \neq$ coeffs of $\chi_G(k)$

  $\neq$ eval. of $\chi_G(k)$

(Cannot distinguish  and )

- $\#$ Chan. in $\mathcal{B}_{tor}(G) \neq T_G(1,0)$

  Nope!

  (Only true if $\mathcal{B}_{tor}(G)$ is $\underline{\text{unimodular}}$)

  [NPS '02]

- In fact, $\#$ Chan. in $\mathcal{B}_{tor}(G)$ is related to the $\underline{\text{bivariate chromatic poly.}}$ $\chi_G(k,l)$

Def: A **proper (k,l)-coloring** of sym. graph

G is a coloring $f: V(G) \to \{0, \pm 1, \cdots \pm k\}$

$\cup \underbrace{\{0_1, 0_2, \cdots, 0_{l-1}\}}_{\text{sym. colors}}$ s.t. [Goodall et.al. '21]

$0_i = -0_i$

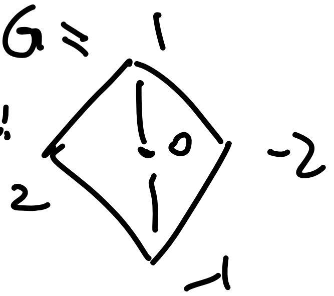
- $f(0) = 0$
- $f(i) = -f(-i)$
- $f(i) \neq f(j)$ if $ij \in E(G)$

$$\#\text{proper } (k,l)\text{-colorings of } G := \underbrace{\chi_G(k,l)}_{\text{bivariate chromatic poly.}}$$

Thm: #chambers in $\mathcal{B}_{tor}(G)$

$$= (-1)^n \chi_G(-1, 2)$$

Ex: $G =$



- If vertex 1 colored by one of $\pm j$

  $2k \cdot (2k+l-2)$

- If vertex 1 colored by 0_i

  $(l-1)(2k+l-1)$

$$\chi_G(k,l) = 2k(2k+l-2) + (l-1)(2k+l-1)$$

$$(-1)^2 \chi_G(-1,2) = \underline{\underline{3}}$$

<u>Proof of Thm</u> (Deletion-Contraction):

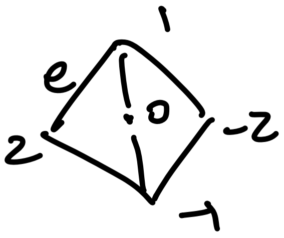
Sym. Graph G. edge $e = i,j \in E(G)$

$-e = -i,-j \in E(G)$

$G \setminus e$: delete $\pm e$

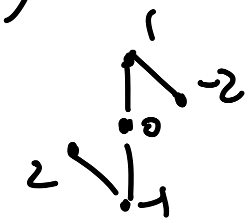
G / e : contract $\pm e$

Ex:

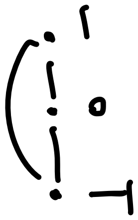


$e=(1,2)$

$G\setminus e =$



$G/e =$



Note: Cannot contract $i, -i$ or $i, 0$.

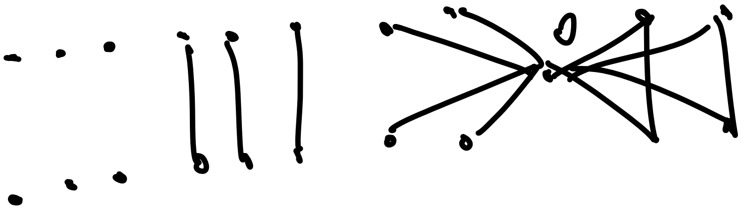
Del-Cont.

$$\#\text{Chan. } B_{tor}(G) = \overbrace{\#\text{Chan. } B_{tor}(G\setminus e)}^{0 \text{ if } G\setminus e \text{ is not weakly conn. anymore}} + \#\text{Chan. } B_{tor}(G/e)$$

$$\chi_G(k,l) = \chi_{G\setminus e}(k,l) - \chi_{G/e}(k,l)$$

Base case

$G =$



Summary

$\#$ Chambers in $\mathcal{B}_{tor}(G)$

$= \#$ Sym. AOs of G $/\sim$

$= (-1)^n \chi_G(-1, 2)$

} Direct proof ? (w/o geometry)

Why do we care ?

- $\mathcal{B}_{tor}(G)$ is the simplist non-trivial non-unimodular toric arr.

  (unimodular case studied in NPS '02)

- Bivariate chromatic poly. is the right thing to think about.

  ($\ell = 0, 1$ case studied in Zas '82)

($\ell=0$ <u>zero-free</u> chrom. poly.

<u>Open</u>: Any combinatorial description when

$\ell=0$, $k\leq 0$?)

- The case of AOs $/\sim$ is fundamentally different in Type A & Type B

In Type A

$T_G(1,0) = \#$ AOs of $G/\sim$

$= \#$AOs of G w/ a unique fixed sink u) represents

Moreover, we can define a poset structure on each equiv. class :

$w \gtrdot w'$ if w w/ source i

↓ flip i

w' w/ sink i

(For any $i \neq u$)

Thm: [Propp '02] This poset is a distributive lattice.

The AO w/ unique sink u is $\hat{1}$.

However, no such nice representative obj. / poset structure in Type B.

Other Qs: If $w \sim w'$, how many flips necessary?

Or, what is the diameter?

Thank You!