A Tale of Three Matroids

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Outline.

1. Graph Rigidity Problem
2. Low-Rank Matrix Completion Problem
3. Tensor Independence Problem
4. Main Theorem & Current Progress

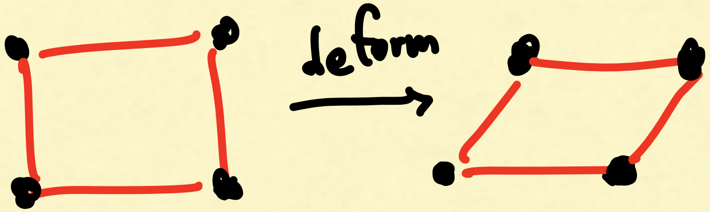
Graph Rigidity Problem.

Def : A d-dim bar-and-joint framework is a graph $G=(V,E)$ with an embedding $p: V(G) \to \mathbb{R}^d$.

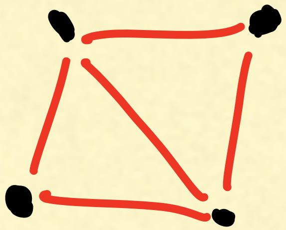
A framework is either rigid or flexible.

Whether the framework can be continuously deformed into a different framework while preserving edge lengths throughout the process.

Ex:



Flexible in $\mathbb{R}^2$

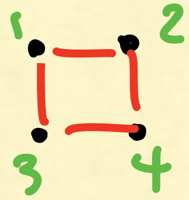


Rigid in $\mathbb{R}^2$

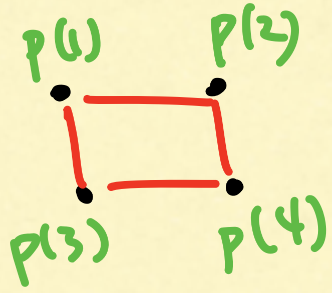
But Flexible in $\mathbb{R}^3$

Generic Rigidity

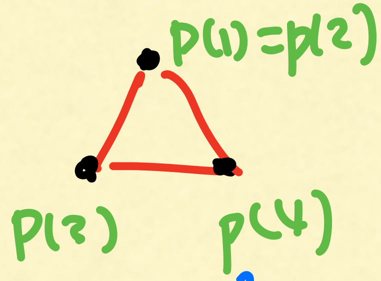
Note: Rigidity might depend on the embedding $p: V(G) \to \mathbb{R}^d$

Ex: $G =$ 

(in $\mathbb{R}^2$)



Flexible



Rigid.

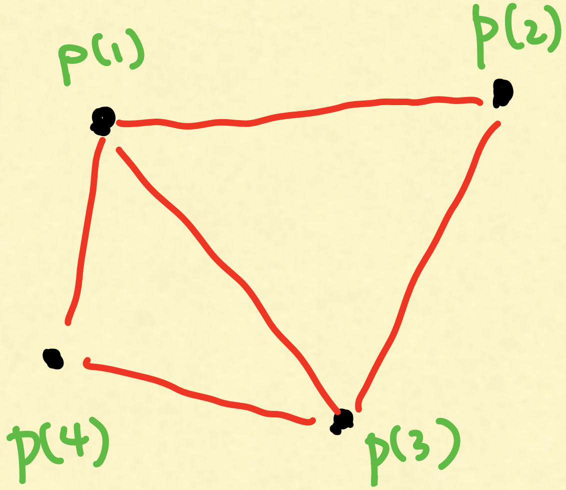
However, if the embedding is "generic", then it only depends on the underlying graph G.

We say G is (generically) rigid in $\mathbb{R}^d$

Generic Rigidity

Def: G is minimally (generically) rigid if G is rigid but becomes flexible after removing any edge.

Ex:

$G =$  is minimally rigid

Prop: Minimally rigid graphs (in $\mathbb{R}^d$) on n vertices form the bases of a matroid $\mathrm{Rigid}_n(d)$

[Whiteley '89
Graver '91]

Matroids

[Oxley '11 Matroid Theory]

Def: A matroid $M=(E,\mathcal{B})$ is a finite set E and a collection of subsets $\mathcal{B} \subset 2^E$ s.t. it satisfies the Basis Exchange Axiom

> if $A, B \in \mathcal{B}$ and $a \in A - B$ then $\exists\, b \in B - A$ s.t. $(A - \{a\}) \cup \{b\} \in \mathcal{B}$

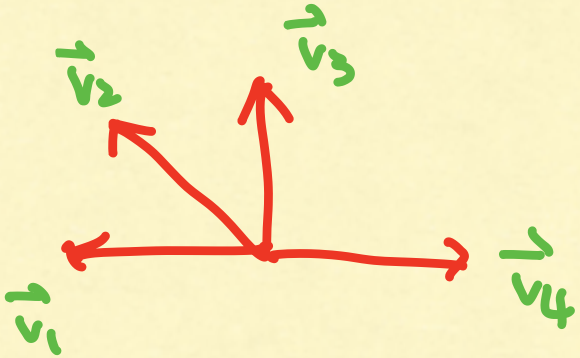
! Matroids is the abstraction of linear independence

Ex: E = collection of vectors $= [4]$

$\mathcal{B}$ = vectors that form a basis

$= \{12, 13, 23, 24, 34\}$.

Rank(M) = size of basis $= 2$.

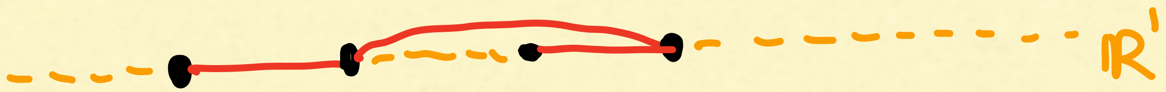


Rigidity Matroid.

<u>Q</u>: What do we know about $\mathrm{Rigid}_n(d)$?

- Rank $= nd - \binom{d+1}{2}$

$d=1$ G is basis $\iff$ G is spanning tree.



$d=2$ "Laman's condition" $\Longrightarrow$

G is a basis iff
- $|E(G)| = 2n-3$
- $|E(G')| \leq 2|V(G')| - 3$ for any subgraph G'

[Pollaczek-Geiringer-Laman]
'27 '70

$d=3$ Open !

Matrix Completion.

Given a $m \times n$ matrix, partially filled w/ "generic" numbers.

Q: Can this partial matrix be completed to rank d?

Ex: $\begin{pmatrix} a & b \\ c & \cdot \end{pmatrix}$ $\quad d=1 \quad$ Yes! Can fill in $\frac{bc}{a}$.

More Examples of Matrix Completion.

Ex:

$$\begin{pmatrix} a & b & \cdot \\ c & \cdot & d \\ \cdot & e & f \end{pmatrix}$$

$d=1$ No! (Hint: consider middle entry)

$d=2$ Yes! (only need 3×3 det vanish)

Fact: Set of (maximally) partially filled $m\times n$ matrices completable to rank d form a matroid $\mathrm{MatComp}_{m,n}(d)$

Tensor Independence

- Consider "generic" vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m \in U \cong \mathbb{R}^{m-a}$

  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in V \cong \mathbb{R}^{n-b}$

- Take tensor products $\vec{u}_i \otimes \vec{v}_j \in U \otimes V \cong \mathbb{R}^{(m-a)(n-b)}$

Q: Which vectors are linearly independent?

$\Longrightarrow$ Linear matroid $\text{Tensor}_{m,n}(a,b)$

<u>Ex</u>: How to take tensor:

$\vec{x} = (x_1, x_2 \cdots x_p) \in \mathbb{R}^p$

$\vec{y} = (y_1, \cdots, y_q) \in \mathbb{R}^q$

$$\vec{x} \otimes \vec{y} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & & \\ x_2 y_1 & x_2 y_2 & \cdots & \\ \vdots & & \ddots & \\ & & & x_p y_q \end{pmatrix} \in \mathbb{R}^{pq}$$

Main Theorem

Punchline : These three problems have the SAME answer. !!

Thm: [BDGGL 24'+] The three matroids

$$\mathrm{BipRigid}_{m,n}(a,b) = \mathrm{MatComp}_{m,n}(a,b) = \left(\mathrm{Tensor}_{m,n}(a,b)\right)^{\vee}$$

are equal up to Matroid Dualities

Denote them as $T_{m,n}(a,b)$.

Matroid Dual:

$M^{\vee} = (E, \mathcal{B}^{\vee})$ where

$\mathcal{B}^{\vee} = \{ E - B \mid B \in \mathcal{B} \}$

Current Progress.

Q. Explicit Description of matroid $T_{m,n}(a,b)$?

a, b small

- $a=b=1$

  Graphical matroid of $K_{m,n}$

- $a=1$, $b>1$

  "Laman-like" conditions

  Rigid [Whiteley '89, GHK+'17] Tensor

- $a=b=2$

  Tropical Geometry.

  [Bernstein '17] MatComp

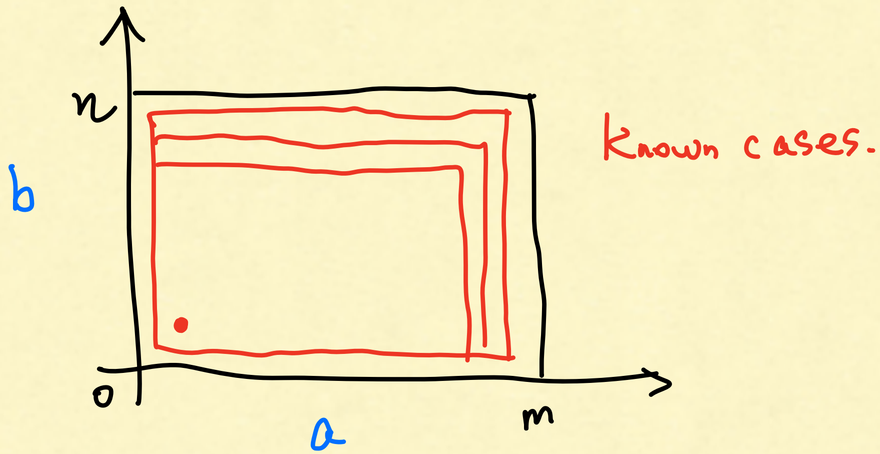
a, b large.

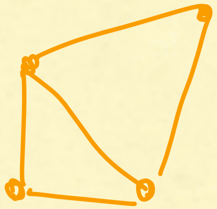
- $m-a \leq 3$ or $n-b \leq 3$

  [BDGGL'24+]

  "Laman-like" conditions

  + one extra condition.





$$\vec{u} \otimes \vec{v} = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & \ddots \\ & \end{pmatrix}$$

Thank You !

$$\begin{pmatrix} a & b & \cdot \\ c & \cdot & d \\ \cdot & e & f \end{pmatrix}$$

A Generalization: Bipartite Rigidity

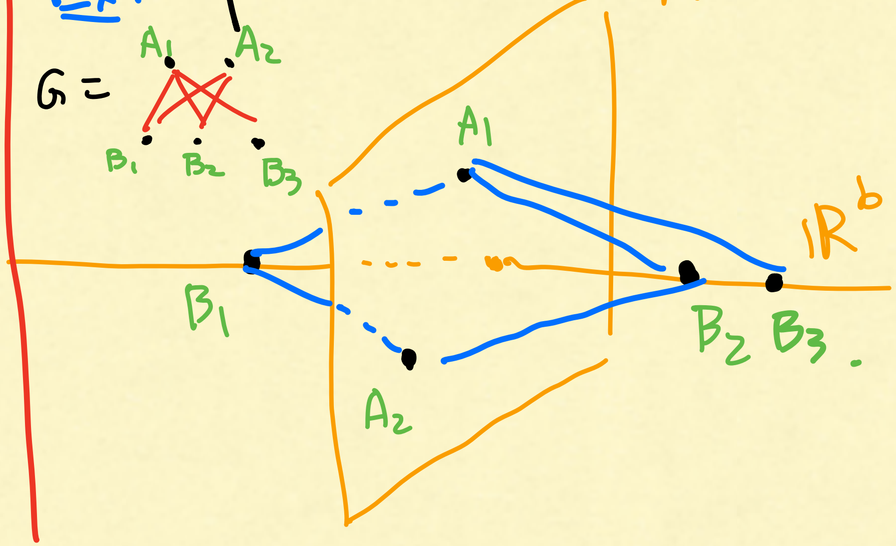
Def: Let $G = (A \cup B, E)$ be a bipartite graph.

[Kalai-Nevo-Novak '16] A **Bipartite embedding** $p : V(G) \to \mathbb{R}^a \oplus \mathbb{R}^b$ is an embedding that respects the bipartite structure. (i.e. p embeds A in $\mathbb{R}^a$ and B in $\mathbb{R}^b$)

Similarly, **rigid** bipartite graphs G on $m+n$ vertices form matroid

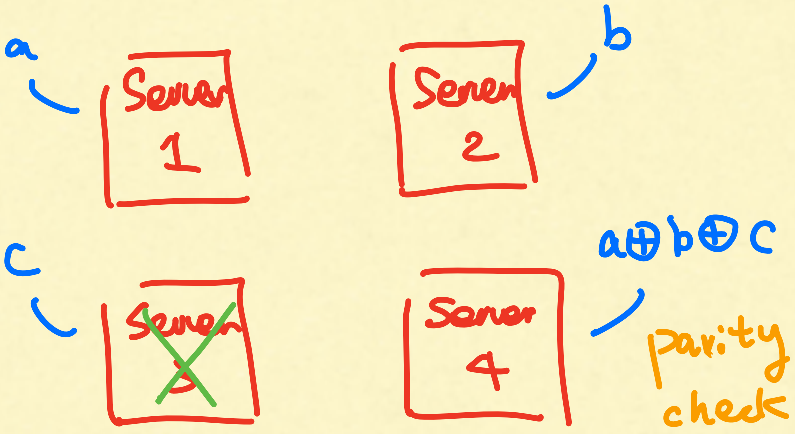
$$\mathrm{BipRigid}_{m,n}(a,b)$$

Ex: Bip. Embed.



Motivation for Tensor Independence

Background: Error correcting codes



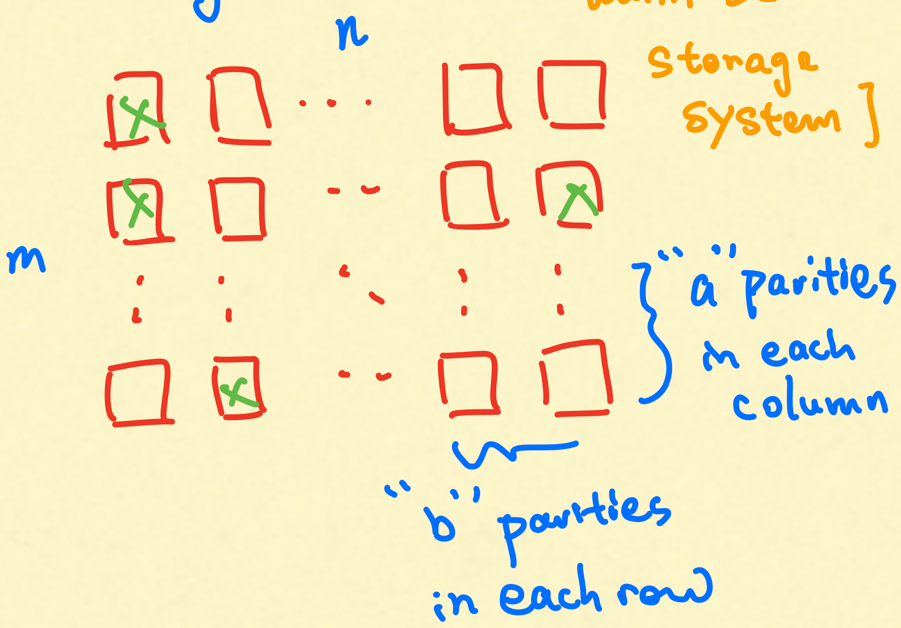
Works w/ 1 failure

Need k parity check servers to tolerate k failures.

Application: Tensor codes

Array of servers.

[f4: facebook's warm BLOB storage system]



Q: Minimally Recoverable Configuration

$\rightarrow$ basis of $\text{Tensor}_{m,n}(a,b)$

Laman-Like Condition

Consider $\mathrm{MatComp}_{m,n}(a,b) = (\Gamma_{m,n}(a,b))^{\vee}$

$$\begin{pmatrix} a & b & & \\ & c & & \\ & d & e & \\ & & & f \end{pmatrix}$$

<u>Fact</u>: Rank tells us #entries $\le na+mb-ab$

The same should hold for all submatrices!

<u>Fact</u>: If a partially filled matrix is completable then any $s \times t$ submatrix have $\le sa+tb-ab$ entries filled

"Laman-like condition"

<u>Thm</u>: [BDGGL'24+]

The Laman-like condition is sufficient

$\iff a \le 1$ or $b \le 1$ or $m-a \le 2$ or $n-b \le 2$

Characteristic Dependence.

The matroids in MatComp and Tensor questions are dependent on characteristic of the base field $\mathbb{F}$.

Why Care: Coding Theorist often work with finite field $\mathbb{F}_p$.

Thm [BDGGL'24] The matroid $T_{m,n}(a,b)$ does NOT depend on the char. p. in all the known case above.

(i.e. $a \leq 1$ or $a=b=2$ or $m-a \leq 3$)

Pf: Only hard one: $a=b=2$. New proof w/o tropical geom.