A Tale of Three Matroids

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Outline.

① Graph Rigidity Problem

② Low-Rank Matrix Completion Problem

③ Tensor Independence Problem

④ Main Theorem

⑤ Current Progress and Conjectures.

Graph Rigidity Problem.

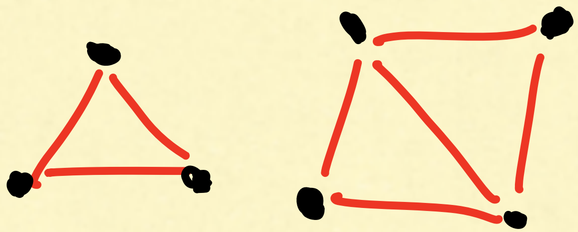
Def : A d-dim bar-and-joint framework is a graph $G=(V,E)$ with an embedding $p: V(G) \to \mathbb{R}^d$.

A framework is either rigid or flexible.

Whether the framework can be continuously deformed into a different framework while preserving edge lengths throughout the process.

Ex: (embeddings into $\mathbb{R}^2$)

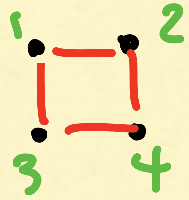
Flexible



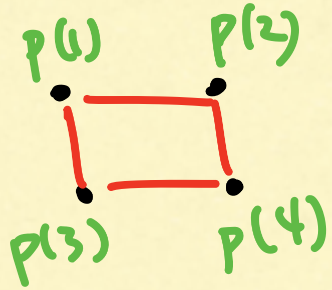
Rigid.

Generic Rigidity

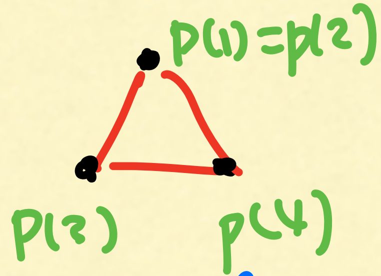
Note: Rigidity might depend on the embedding $p: V(G) \to \mathbb{R}^d$

Ex: $G=$ 

(in $\mathbb{R}^2$)



Flexible



Rigid.

However, if the embedding is "generic", then it only depends on the underlying graph G.

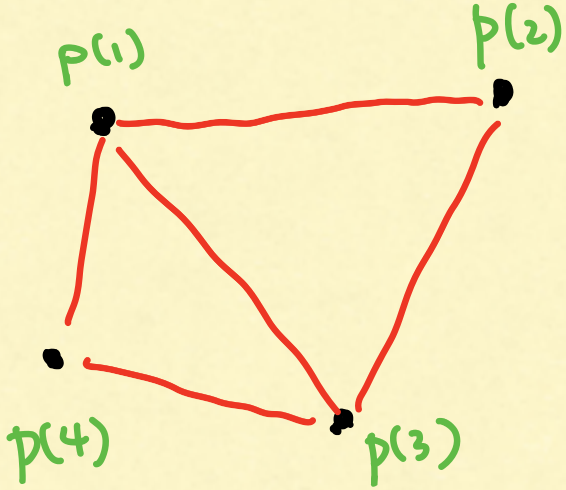
We say G is (generically) rigid in $\mathbb{R}^d$

Generic Rigidity

Def: G is minimally (generically) rigid if G is rigid

but becomes flexible after removing any edge.

Ex:

$G =$  is minimally rigid.

Prop: Minimally rigid graphs (in $\mathbb{R}^d$) on n vertices

[Whiteley '89

Graver '91] form the bases of a matroid $\mathrm{Rigid}_n(d)$

Matroids

[Oxley '11 Matroid Theory]

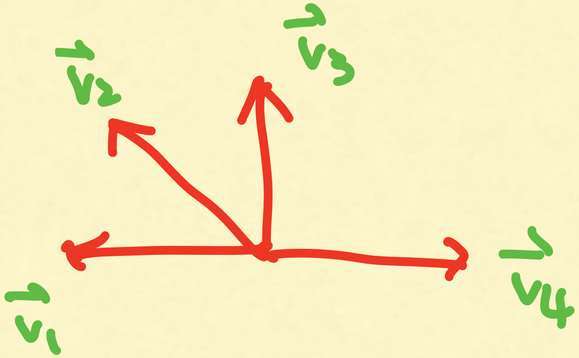
Def: A matroid $M = (E, \mathcal{B})$ is a finite set E and a collection of subsets $\mathcal{B} \subset 2^E$ s.t. it satisfies the Basis Exchange Axiom

if $A, B \in \mathcal{B}$ and $a \in A - B$ then $\exists$ $b \in B - A$ s.t. $(A - \{a\}) \cup \{b\} \in \mathcal{B}$

Ex: E = collection of vectors = $[4]$

$\mathcal{B}$ = vectors that form a basis

$= \{12, 13, 23, 24, 34\}$.



Rank (M) = size of basis = 2

Indep. sets $\mathcal{I} = \{\phi, 1, 2, 3, 4,$

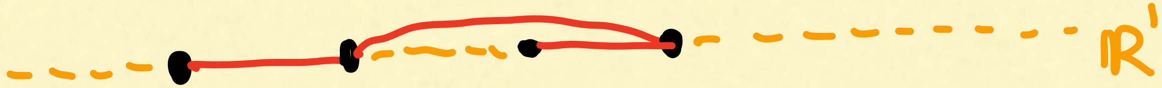
$12, 13, 23, 24, 34\}$

Rigidity Matroid.

Q: What do we know about $\text{Rigid}_n(d)$?

- Rank $= nd - \binom{d+1}{2}$

$d=1$ G is basis $\Longleftrightarrow$ G is spanning tree.



$d=2$ "Laman's condition" $\Longrightarrow$

[Pollaczek-Geiringer-Laman]
'27 '70

$\boxed{\begin{array}{l} G \text{ is a basis iff} \\ \bullet\ |E(G)| = 2n-3 \\ \bullet\ |E(G')| \leq 2|V(G')|-3 \\ \quad \text{for any subgraph } G' \end{array}}$

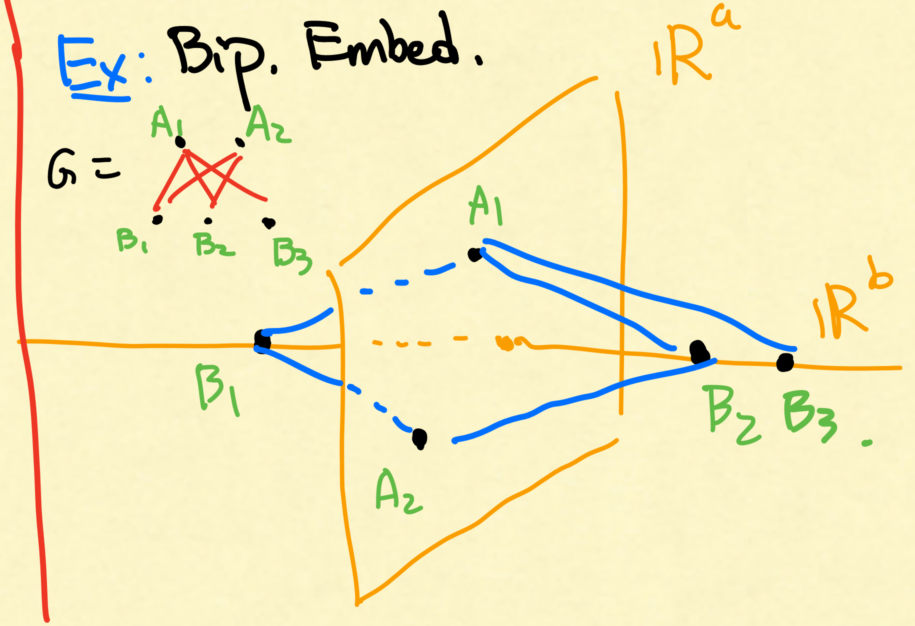
$d=3$ Open !

A Generalization: Bipartite Rigidity

Def: Let $G=(A\cup B, E)$ be a bipartite graph.

[Kalai-Nevo-Novak '16] A **Bipartite embedding** $p: V(G) \to \mathbb{R}^a \oplus \mathbb{R}^b$ is an embedding that respects the bipartite structure. (i.e. p embeds A in $\mathbb{R}^a$ and B in $\mathbb{R}^b$)

Ex: Bip. Embed.



Similarly, **rigid** bipartite graphs G.

$\Longrightarrow$ matroid $\mathrm{BipRigid}_{\underset{|A|=m}{m},\,\underset{|B|=n}{n}}(a,b)$

Matrix Completion.

(in $\mathbb{C}$)

Given a $m \times n$ matrix, partially filled w/ "generic" numbers.

<u>Q</u>: Can this partial matrix be completed to rank d?

<u>Ex</u>: $\begin{pmatrix} a & b \\ c & \cdot \end{pmatrix}$ $\quad d=1 \quad$ Yes! Can fill in $\frac{bc}{a}$.

(No need to consider edge case $a=0$ since "generic")

More Examples of Matrix Completion.

Ex:

$$\begin{pmatrix} a & b & \cdot \\ c & \cdot & d \\ \cdot & e & f \end{pmatrix}$$

$d=1$ No! (Hint: consider middle entry)

$d=2$ Yes! (only need 3x3 det vanish)

Fact: Set of partially filled $m \times n$ matrices that can be completed to rank d form the indep. sets of a matroid!

subsets of bases

$\mathrm{MatComp}_{m,n}(d)$

Generalizations of Matrix Completion.

Matrix	Matroid.
$m\times n$ matrix rank d. $\begin{pmatrix} a & b & \cdot \\ c & \cdot & d \\ \cdot & e & f \end{pmatrix}$	$\mathrm{MatComp}_{m,n}(d)$ $E=[m]\times[n]$ $\quad$ $\mathrm{rank}=md+nd-d^2$
$n\times n$ symmetric matrix rank d. $\begin{pmatrix} a & b & \cdot \\ b & d & c \\ \cdot & c & \cdot \end{pmatrix}$	$\mathrm{SymComp}_n(d)$ $E=\binom{[n]}{2}+[n]$ $\quad$ $\mathrm{rank}=nd-\binom{d}{2}$
$n\times n$ skew-sym matrix rank $2d$ $\begin{pmatrix} 0 & a & b \\ -a & 0 & \cdot \\ -b & \cdot & 0 \end{pmatrix}$	$\mathrm{SkewComp}_n(2d)$ $E=\binom{[n]}{2}$ $\quad$ $\mathrm{rank}=2nd-\binom{2d+1}{2}$

Tensor Independence

- Consider "generic" vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m \in U \cong \mathbb{R}^{m-a}$

  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in V \cong \mathbb{R}^{n-b}$

- Take tensor products $\vec{u}_i \otimes \vec{v}_j \in U \otimes V \cong \mathbb{R}^{(m-a)(n-b)}$

Q: Which vectors are linearly independent?

$\Longrightarrow$ Linear matroid $\text{Tensor}_{m,n}(a,b)$

Ex: How to take tensor:

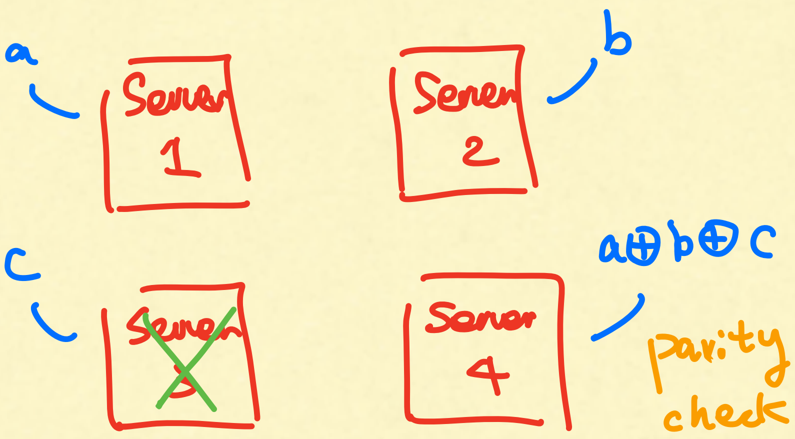
$\vec{x} = (x_1, x_2 \cdots x_p) \in \mathbb{R}^p$

$\vec{y} = (y_1, \cdots, y_q) \in \mathbb{R}^q$

$$\vec{x} \otimes \vec{y} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & & \\ x_2 y_1 & x_2 y_2 & \ddots & \\ \vdots & & \ddots & \\ & & & x_p y_q \end{pmatrix} \in \mathbb{R}^{pq}$$

Motivation for Tensor Independence

Background: Error correcting codes



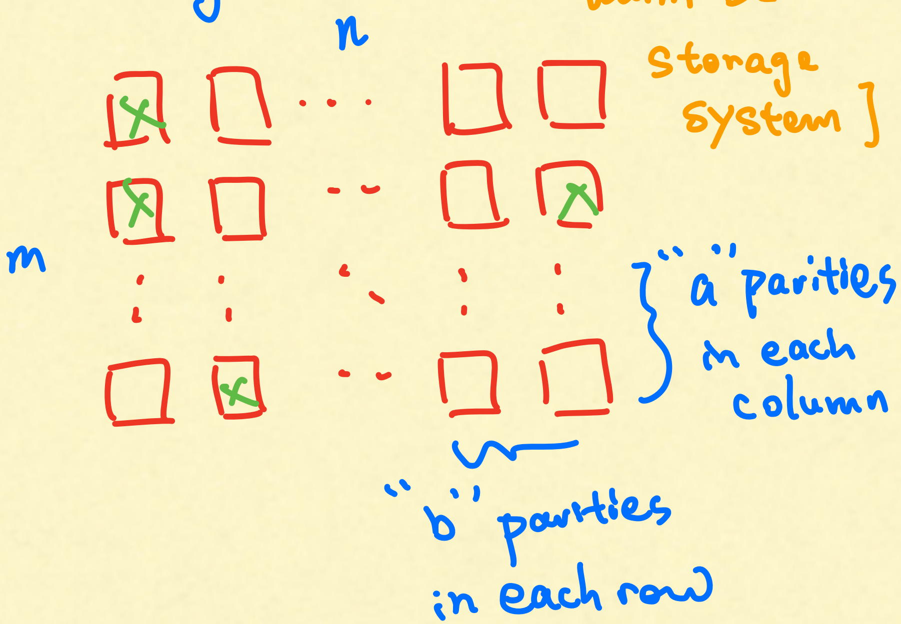
Works w/ 1 failure

Need k parity check servers to tolerate k failures.

Application: Tensor codes

Array of servers.

[f4: facebook's warm BLOB storage system]



Q: Minimally Recoverable Configuration

→ basis of $\text{Tensor}_{m,n}(a,b)$

Generalizations of Tensor Independence

Tensor	Matroid.
Tensor product of mn vectors. $\vec{u}_i \otimes \vec{v}_j \in U_{m-a} \otimes V_{n-b}$	$\mathrm{Tensor}_{m,n}(a,b)$ $E = [m] \times [n]$ rank $= (m-a)(n-b)$
Sym. Power of n vectors. $\vec{v}_i \otimes \vec{v}_j \in \mathrm{Sym}^2 V_{n-d}$	$\mathrm{SymTensor}_n(d)$ $E = \binom{[n]}{2} + [n]$ rank $= \binom{n-d+1}{2}$
Wedge of n vectors. $\vec{v}_i \wedge \vec{v}_j \in \Lambda^2 V_{n-d}$	$\mathrm{WedgeTensor}_n(d)$ $E = \binom{[n]}{2}$ rank $= \binom{n-d}{2}$

Main Theorem

Punchline : These three problems have the SAME answer. !!

Thm: [BDGGL 24'+] The three matroids

$\mathrm{BipRigid}_{m,n}(a,b)$ $\mathrm{MatComp}_{m,n}(d)$ $\mathrm{Tensor}_{m,n}(a,b)$

are the same up to basic matroid operations.

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① Matroid Dual

$M^\vee = (E, \mathcal{B}^\vee)$

$\mathcal{B}^\vee = \{E - B : B \in \mathcal{B}\}$

② Matroid Deletion

$M - T = (E - T, \mathcal{B} - T)$

$\mathcal{B} - T = \{B : B \in \mathcal{B}, B \cap T = \phi\}$.

Main Theorem

	Bipartite	Symmetric	Anti-Sym.
Tensor	$\mathrm{Tensor}_{m,n}(a,b) = T_{m,n}(a,b)$	$\mathrm{SymTensor}_n(d) = S_n(d)$	$\mathrm{WedgeTensor}_n(d) = W_n(d)$
Graph Rigidity	$\mathrm{BipRigid}_{m,n}(a,b) = (T_{m,n}(a,b))^\vee$	$\mathrm{Rigid}_n(d) = (S_n(d+1) - \mathrm{diag})^\vee$	/
Matrix Completion	$\mathrm{MatComp}_{m,n}(d) = (T_{m,n}(d,d))^\vee$	$\mathrm{SymComp}_n(d) = (S_n(d))^\vee$	$\mathrm{SkewComp}_n(2d) = (W_n(2d))^\vee$

Current Progress.

Q. Explicit Description of matroid $T_{m,n}(a,b)$?

a,b small

- $a=b=1$

  Graphical matroid of $K_{m,n}$

- $a=1$, $b>1$

  "Laman-like" conditions

  rigidity [Whiteley '89, GHK+'17] tensor code

- $a=b=2$

  Tropical Geometry.

  [Bernstein '17] matrix compl.

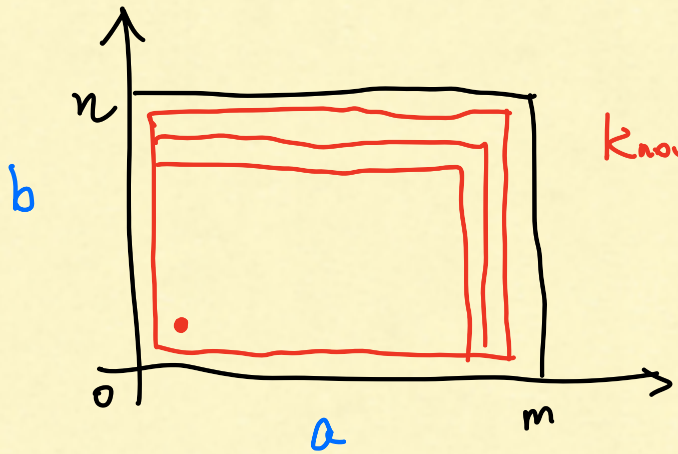
a,b large.

- $m-a\le 3$ or $n-b\le 3$

  [BDGGL'24 +]

  "Laman-like" conditions

  + one extra condition.



known cases.

Characteristic Dependence.

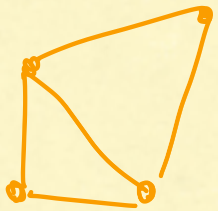
The matroids in MatComp and Tensor questions are dependent on characteristic of the base field $\mathbb{F}$.

Why Care : Coding Theorist often work with finite field $\mathbb{F}_p$.

Thm [BDGGL'24] The matroid $T_{m,n}(a,b)$ does NOT depend on the char. p. in all the known case above.

(i.e. $a \le 1$ or $a=b=2$ or $m-a \le 3$)

Pf: Only hard one : $a=b=2$. New proof w/o tropical geom.



$$\vec{u} \otimes \vec{v} = \begin{pmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & \ddots \end{pmatrix}$$

Thank You !

$$\begin{pmatrix} a & b & \cdot \\ c & \cdot & d \\ \cdot & e & f \end{pmatrix}$$

* Special Thanks to Alex Postnikov and Yibo Gao for introducing us to the matrix completion problem.

Laman-Like Condition

Consider $\mathrm{MatComp}_{m,n}(a,b) = (\Gamma_{m,n}(a,b))^{\vee}$

$$\begin{pmatrix} a & b & & \\ & c & & \\ & d & e & \\ & & & f \end{pmatrix}$$

Fact: Rank tells us #entries $\leq na+mb-ab$

The same should hold for all submatrices!

Fact: If a partially filled matrix is completable then any submatrix have $\leq ma+nb-ab$ entries filled

"Laman-like condition"

Thm: [BDGGL'24+]

The Laman-like condition is sufficient

$\iff a\leq 1$ or $b \leq 1$ or $m-a\leq 2$ or $n-b\leq 2$