Quantum Bruhat graphs and tilted Richardson varieties

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Outline

- Quantum Bruhat Graphs
- Tilted Richardson varieties
- Tilted Deodhar decomposition

Cohomology of Flag Varieties

Let $\mathrm{Fl}_n = \mathrm{GL}_n(\mathbb{C})/B$ be the **complete flag variety** over $\mathbb{C}$.

The cohomology ring $H^*(\mathrm{Fl}_n)$ is a free $\mathbb{Z}$-module generated by Schubert classes $\{\sigma_w : w \in S_n\}$, where σ_w is the cohomology class of the Schubert variety X_w.

$$\sigma_u \cdot \sigma_v = \sum_{w \in S_n} c_{u,v}^w \sigma_w$$

Open problem

Find a combinatorial interpretation for the LR coefficients $c_{u,v}^w$.

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Quantum Cohomology of Flag Varieties

The **quantum cohomology ring** $QH^*(\mathrm{Fl}_n) \cong H^*(\mathrm{Fl}_n) \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, \dots, q_{n-1}]$ is a free $\mathbb{Z}[q]$-module generated by the Schubert classes $\{\sigma_w : w \in S_n\}$.

$$\sigma_u \star \sigma_v = \sum_{w,d} c_{u,v}^{w,d}\, q^d \sigma_w$$

Here $d = (d_1, \dots, d_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$ and $q^d := q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$.

For example, in $QH^*(\mathrm{Fl}_3)$,

$$\sigma_{213} \star \sigma_{321} = q_1 \cdot \sigma_{231} + q_1 q_2 \cdot \sigma_{123}$$

Motivating Q: What weights q^d appear in the quantum product $\sigma_u \star \sigma_v$? What is the minimal such q^d?
[Fulton-Woodward '04, Postnikov '05, Buch-Chung-Li-Mihalcea '20]

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Quantum Bruhat Graph

Postnikov ('04) related all quantum degrees appearing in $\sigma_u \star \sigma_{w_0 v}$ to weights of paths on the quantum Bruhat graph.

Definition (Brenti-Fomin-Postnikov '99)

The **quantum Bruhat graph** Γ_n is a weighted directed graph on S_n with the following two types of edges:

$$\begin{cases} w \to wt_{ij} \text{ of weight } 1 & \text{if } \ell(wt_{ij}) = \ell(w) + 1, \\ w \to wt_{ij} \text{ of weight } q_{ij} := q_i \cdots q_{j-1} & \text{if } \ell(wt_{ij}) = \ell(w) - \ell(t_{ij}) \end{cases}$$

Theorem (Postnikov '04)

There is a unique minimal q^d that appears in $\sigma_u \star \sigma_{w_0 v}$. Such $q^{d_{min}}$ is the weight of any shortest path $u \to v$ in Γ_n.

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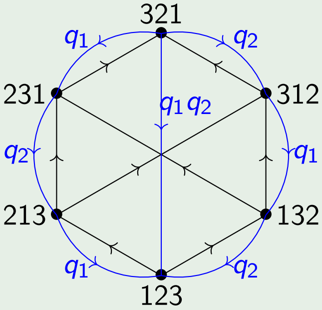
Quantum Bruhat Graph

Example (Quantum Bruhat graph Γ_3)



For $u = 231$, $v = 123$, there are two shortest paths, both of which have minimal weight $q^{d_{\min}} = q_1 q_2$. This is also the minimal q^d that appears in $\sigma_u \star \sigma_{w_0 v}$.

A Simple Formula for $q^{d_{\min}}$

Theorem (G.-Gao-Gao '23)

For $u, v \in S_n$, the weight of any shortest path $u \to v$ is $q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$, where d_k = depth of the lattice path $P(u[k], v[k])$ formed by using $u[k] := \{u_1, \ldots, u_k\}$ as upsteps and $v[k] := \{v_1, \ldots, v_k\}$ as downsteps.

Example

For $u = 4637521$, $v = 5312467$ and $k = 4$, upsteps $u[k] = \{3, 4, 6, 7\}$ and downsteps $v[k] = \{1, 2, 3, 5\}$. We have $d_k = \text{depth} = 2$.

$P(u[k], v[k]) =$

$\uparrow$ depth $= 2$ $\downarrow$

In total, $q^{d_{\min}} = q_2 q_3 q_4^2 q_5^2 q_6$.

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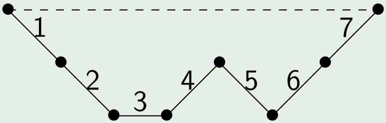

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Tilted Bruhat Interval

Tilted bruhat intervals are "quantum analogs" of (strong) Bruhat intervals.

Definition (Brenti-Fomin-Postnikov '99)

For $u, v \in S_n$, the **tilted Bruhat interval** $[u, v]$ is a partial order on the set

$$[u, v] := \{w \in S_n : \exists \text{ shortest path } u \to v \text{ passing through } w \text{ in } \Gamma_n\}$$

where $w \le w'$ iff there exists a shortest path $u \to v$ passing through w then w' (i.e. $u \to w \to w' \to v$).

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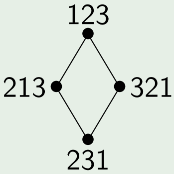

Curve Neighbourhoods (alert: geometry!!)

Definition (Buch-Chaput-Mihalcea-Perrin '13)

For $u, v \in S_n$, and degree $d = (d_1, \ldots, d_{n-1})$ the **two-pointed curve neighborhood** $\Gamma_d(X^u, X_v)$ is the union of all degree d rational curves that passes through both Schubert varieties X^u and X_v in Fl_n.

When q^d appears in $\sigma_u \star \sigma_{w_0 v}$, its cohomology class encodes information about the quantum product:

$$[\Gamma_d(X^u, X_v)] = \frac{1}{c}[q^d]\sigma_u \star \sigma_{w_0 v} \in H^*(\mathrm{Fl}_n),$$

for some $c \in \mathbb{Z}_{>0}$.

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Curve Neighbourhoods

- In the Grassmannian, the curve neighborhoods are positroid varieties [Knutson-Lam-Speyer '13];
- In cominuscule G/P, they are projected Richardson varieties [Buch-Chaput-Mihalcea-Perrin '18].

In the complete flag variety,

- If $d = (0, \dots, 0)$, $\Gamma_d(X^u, X_v) = \mathcal{R}_{u,v} (:= X^u \cap X_v)$ is the Richardson variety;
- If $d = (0, \dots, 0, 1, 0, \dots)$, then $\Gamma_d(X^u, X_v)$ is a Richardson variety [L-M'13];
- If $u = \text{id}$, then $\Gamma_d(X^u, X_v) = X_{v(d)}$ is a Schubert variety [B-M'15];
- $\Gamma_d(X^u, X_v)$ is empty unless $d \geq d_{\min}$ coordinate-wise [F-W'04].

In general, we don't know which flags $gB \in \Gamma_d(X^u, X_v)$.
Goal: Give a concrete description when $d = d_{min}$ in terms of rank conditions.

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Shifted Gale order

For $a \in [n]$, let $\leq_a$ be the total order of $[n]$ where

$$a <_a a+1 <_a \cdots <_a n <_a 1 <_a \cdots <_a a-1.$$

For $I = \{i_1 <_a \cdots <_a i_k\}, J = \{j_1 <_a \cdots <_a j_k\} \in \binom{[n]}{k}$, we say $I \leq_a J$ if

$$i_m \leq_a j_m \quad \text{for all } m \in [k].$$

Fact

For any permutation u, v, there exists a sequence $\mathbf{a} = (a_1, \ldots, a_{n-1})$ such that

$$u[k] := \{u_1, \ldots, u_k\} \leq_{a_k} v[k] \text{ for all } k \in [n-1].$$

In this case, we write $u \leq_{\mathbf{a}} v$.

Example

Let $u = 4321, v = 3142$. We can set $\mathbf{a} = (4, 2, 2)$ so that $u \leq_{\mathbf{a}} v$.

$$\{4\} \leq_4 \{3\}, \{3, 4\} \leq_2 \{3, 1\} \text{ and } \{2, 3, 4\} \leq_2 \{3, 4, 1\}.$$

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Tilted Richardson Varieties

For $u, v \in S_n$ and a sequence $\mathbf{a}$ such that $u \leq_{\mathbf{a}} v$, we define the **tilted Richardson variety/cell** $\mathcal{T}_{u,v,\mathbf{a}}$ (and $\mathcal{T}^{\circ}_{u,v,\mathbf{a}}$) by some "rank conditions".

Example

Consider $u = 4321(\star)$, $v = 3142(\bullet)$, and $\mathbf{a} = (4, 2, 2)$ such that $u \leq_{\mathbf{a}} v$.
Rank conditions of matrices in $\mathcal{T}_{u,v,\mathbf{a}}$ in the left $k = 2$ columns:

$M =$

			$\star$	1
		$\star$		2
	$\star$			3
$\star$				4

There are 6 rank conditions for each $k \in \{1, 2, 3\}$.
Replacing "$\leq$" with "$=$" gives us rank conditions for $\mathcal{T}^{\circ}_{u,v,\mathbf{a}}$.

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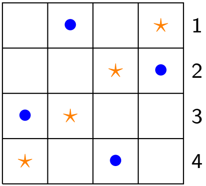

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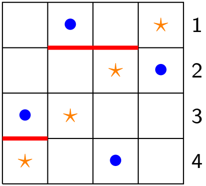

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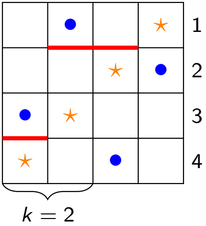

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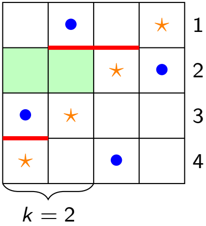
$M =$ 

rank of green
$\leq \#\star$'s in green $= 0$

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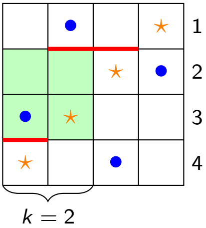
For $u, v \in S_n$ and a sequence $\mathbf{a}$ such that $u \leq_{\mathbf{a}} v$, we define the **tilted Richardson variety/cell** $\mathcal{T}_{u,v,\mathbf{a}}$ (and $\mathcal{T}^{\circ}_{u,v,\mathbf{a}}$) by some "rank conditions".

Example

Consider $u = 4321(\star)$, $v = 3142(\bullet)$, and $\mathbf{a} = (4, 2, 2)$ such that $u \leq_{\mathbf{a}} v$.
Rank conditions of matrices in $\mathcal{T}_{u,v,\mathbf{a}}$ in the left $k = 2$ columns:

Start from the row below red line and go down:

$M =$



rank of green
$\leq \#\star$'s in green $= 1$

There are 6 rank conditions for each $k \in \{1, 2, 3\}$.
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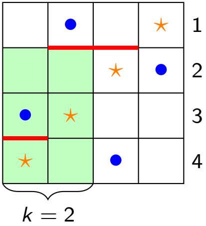

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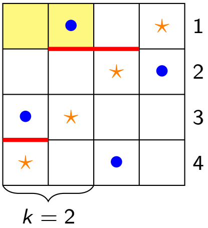

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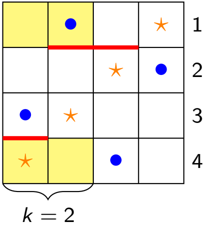
$M =$ 

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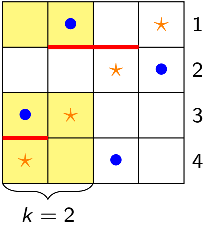

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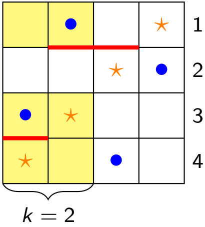

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Theorem (G.-Gao-Gao '23)

$\mathcal{T}_{u,v,\mathbf{a}}$ *does not depend on the choice of* $\mathbf{a}$. *So we denote this space as* $\mathcal{T}_{u,v}$.
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Special cases:

- If $u = \mathrm{id}$, $\mathcal{T}_{\mathrm{id},v} = X_v$ is the Schubert variety;
- If $v = w_0$, $\mathcal{T}_{u,w_0} = X^u$ is the opposite Schubert variety;
- If $u \leq v$, $\mathcal{T}_{u,v} = \mathcal{R}_{u,v} (:= X^u \cap X_v)$ is the Richardson variety, $\mathcal{T}^{\circ}_{u,v} = \mathcal{R}^{\circ}_{u,v}$ is the Richardson cell.

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Main Theorem

Theorem (G.-Gao-Gao '23 '23+)

- *$\mathcal{T}_{u,v}$ is a closed subvariety of Fl_n, and $\mathcal{T}^\circ_{u,v}$ is open in $\mathcal{T}_{u,v}$,*
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Deodhar decomposition

Motivation: Given a pair of permutations $u \le v$, the **Kazhdan-Lusztig R polynomial** is defined as

$$R_{u,v}(q) := \#\mathcal{R}^{\circ}_{u,v}(\mathbb{F}_q).$$

They are elementary pieces used to construct the **Kazhdan-Lusztig polynomials** $P_{u,v}(q)$, which plays a fundamental role in representation theory.

Main idea: In order to understand $R_{u,v}(q)$, Deodhar ('85) introduced **Deodhar decomposition**, which decomposes the Richardson cell $\mathcal{R}^{\circ}_{u,v}$ into **simple pieces** that are isomorphic to $\mathbb{C}^a \times (\mathbb{C}^*)^b$.

$$\mathcal{R}^{\circ}_{u,v} = \bigsqcup_{\alpha} \mathbb{C}^a \times (\mathbb{C}^*)^b \implies R_{u,v}(q) = \sum_{\alpha} q^a (q-1)^b.$$

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Deodhar decomposition

The "simple pieces" are labeled by the following data:

Definition (Marsh-Rietsch '03)

Fix a reduced word $\mathbf{v} = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of v. A **distinguished subword** for u is $\mathbf{u} = u_1 \cdots u_\ell$ where each $u_k \in \{1, s_{i_k}, s_{i_k}\}$ such that

$$u_k = \begin{cases} 1 \text{ or } s_{i_k}, & \text{if } \ell(u_1 \cdots u_{k-1}) < \ell(u_1 \cdots u_{k-1} s_{i_k}), \\ s_{i_k}, & \text{if } \ell(u_1 \cdots u_{k-1}) > \ell(u_1 \cdots u_{k-1} s_{i_k}). \end{cases}$$

and their product is u. We denote $\mathbf{u} \prec \mathbf{v}$.

Example

If $\mathbf{v} = s_1 s_2 s_1$, there are two distinguished subwords for $u = \mathrm{id}$:

$$\mathbf{u} = 111 \qquad \mathbf{u} = s_1 1 s_1$$

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The Richardson cell $\mathcal{R}^\circ_{u,v}$ can be decomposed into **Deodhar cells** $D_{\mathbf{u},\mathbf{v}}$:

Theorem (Deodhar '85, Marsh-Rietsch '03)

$$\mathcal{R}^\circ_{u,v} = \bigsqcup_{\mathbf{u}\prec\mathbf{v}} D_{\mathbf{u},\mathbf{v}}, \text{ where each } D_{\mathbf{u},\mathbf{v}} \cong (\mathbb{C}^*)^{\#1\text{'s in } \mathbf{u}} \times \mathbb{C}^{\# s_{i_k}\text{'s in } \mathbf{u}}$$

Each Deodhar cell $D_{\mathbf{u},\mathbf{v}}$ can be explicitly parametrized by products of simple matrices $y_i(p)$, $x_i(m)$ and $\dot{s}_i$.

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Therefore, $\mathcal{R}^\circ_{123,321} = D_{111,s_1s_2s_1} \sqcup D_{s_1 1 s_1, s_1 s_2 s_1} \cong (\mathbb{C}^*)^3 \sqcup (\mathbb{C}^* \times \mathbb{C})$.

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Example of Deodhar decomposition

$$\mathcal{R}^{\circ}_{123,321} = D_{111,s_1s_2s_1} \sqcup D_{s_11s_1,s_1s_2s_1} \cong (\mathbb{C}^*)^3 \sqcup (\mathbb{C}^* \times \mathbb{C}).$$

$$D_{111,s_1s_2s_1} = y_1(p_1)y_2(p_2)y_1(p_3)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ p_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ p_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ p_1+p_3 & 1 & 0 \\ p_2p_3 & p_2 & 1 \end{pmatrix} \cong (\mathbb{C}^*)^3.$$

$$D_{s_11s_1,s_1s_2s_1} = \dot{s}_1 y_2(p_2) x_1(m_3) \dot{s}_1^{-1}$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p_2 & 1 \end{pmatrix} \begin{pmatrix} -m_3 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -m_3 & 1 & 0 \\ -p_2 & 0 & 1 \end{pmatrix} \cong \mathbb{C}^* \times \mathbb{C}.$$

Upshot: One can write down Deodhar cells explicitly!

Tilted Deodhar decomposition

Definition (G.-Gao-Gao '23+)

Fix a tilted reduced word $\mathbf{v} = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of v. A **tilted distinguished subword** for u is $\mathbf{u} = u_1 \cdots u_\ell$ where each $u_k \in \{1, s_{i_k}, s_{i_k}, s_{i_k}\}$ that follows certain rules and their product is u. We denote $\mathbf{u} \prec_t \mathbf{v}$.

Example

For $u = 512346$ and $v = 246513$, given a tilted reduced word $\mathbf{v} = s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_3 s_2 s_1 s_1 s_2$, there are 4 tilted distinguished subwords:

$\mathbf{u} = 111111 s_4 s_3 s_2 s_1 11$ $\qquad$ $\mathbf{u} = 111 s_1 11 s_4 s_3 s_2 s_1 1 s_2$

$\mathbf{u} = s_3 1111 s_3 s_4 s_3 s_2 s_1 11$ $\qquad$ $\mathbf{u} = s_3 11 s_1 1 s_3 s_4 s_3 s_2 s_1 1 s_2$

Theorem (G.-Gao-Gao '23+)

$$\mathcal{T}^\circ_{u,v} = \bigsqcup_{\mathbf{u} \prec_t \mathbf{v}} D_{\mathbf{u},\mathbf{v}}, \ \ \textit{where each } D_{\mathbf{u},\mathbf{v}} \cong (\mathbb{C}^*)^{\#1\text{'s in } \mathbf{u}} \times \mathbb{C}^{\#s_{i_k}\text{'s in } \mathbf{u}}$$

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Why tilted Deodhar?

Reason 1: Proof of irreducibility of $\mathcal{T}_{u,v}$

Theorem (G.-Gao-Gao '23+)

For any tilted reduced word $\mathbf{v}$*, there exists a unique* **positive distinguished subword** $\mathbf{u}_+ \prec_t \mathbf{v}$ *that does not use* s_{i_k}*. The corresponding tilted Deodhar cell* $D_{\mathbf{u}_+,\mathbf{v}} \cong (\mathbb{C}^*)^{\ell(u,v)}$ *is the* **unique cell of maximal dimension***, and all other cells* $D_{\mathbf{u},\mathbf{v}}$ *have dimension* $< \ell(u,v)$*.*

Theorem above combined with $\dim(\mathcal{T}_{u,v}) = \ell(u,v)$ implies

Corollary (G.-Gao-Gao '23+)

$\mathcal{T}_{u,v}$ *is irreducible, and* $D_{\mathbf{u}_+,\mathbf{v}} \subset \mathcal{T}_{u,v}$ *is a dense subset.*

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Reason 2: Total Positivity

Theorem (Björner '84, Dyer '93)

If a poset is thin and EL-shellable, then the poset is the face poset of a regular CW-complex.

Theorem (Brenti-Fomin-Postnikov '99)

The tilted Bruhat interval $[u, v]$ is thin and EL-shellable.

Motivating Q: Can one find a natural realization of such CW-complex?

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For a sequence $\mathbf{a} = (a_1, \ldots, a_{n-1}) \in [n]^{n-1}$, the **tilted totally nonnegative flag variety** $\mathrm{TNN}_{\mathbf{a}} \subset \mathrm{Fl}_n(\mathbb{R})$ is the set of all flags gB whose Plücker coordinates

$$\Delta_{i_1,\ldots,i_k}(g) \geq 0, \text{ for any } i_1 <_{a_k} < i_2 <_{a_k} \cdots <_{a_k} i_k, \forall k$$

under the shifted total order $<_{a_k}$.

Example

If $n = 4$, $\mathbf{a} = (4, 3, 2)$, then flags in $\mathrm{TNN}_{\mathbf{a}}$ satisfies:

$$\Delta_4, \Delta_1, \Delta_2, \Delta_3 \geq 0$$

$$\Delta_{34}, \Delta_{31}, \Delta_{32}, \Delta_{41}, \Delta_{42}, \Delta_{12} \geq 0$$

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Definition (G.-Gao-Gao '23+)

Define the **TNN part of tilted Richardson cell/variety** to be $\mathcal{T}_{u,v}^{\circ,\geq 0} := \mathcal{T}_{u,v}^{\circ} \cap \mathrm{TNN}_{\mathbf{a}}$ and $\mathcal{T}_{u,v}^{\geq 0} := \mathcal{T}_{u,v} \cap \mathrm{TNN}_{\mathbf{a}}$ for any **a** such that $u \leq_{\mathbf{a}} v$. This definition is independent of the choice of **a**.

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Why tilted Deodhar?

Reason 3: Tilted R polynomial

Given a pair of permutations, the tilted R polynomial is defined as

$$R^{\text{tilt}}_{u,v}(q) := \#\mathcal{T}^{\circ}_{u,v}(\mathbb{F}_q).$$

We can use tilted Deodhar decomposition to compute these polynomials.

Combinatorial Invariance Problem (Lusztig '83, Dyer '87)

The Kazhdan-Lusztig R polynomial $R_{u,v}(q)$ depends only on the poset structure of the strong Bruhat interval $[u, v]$.

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Thank you all for listening!

Part 1: arXiv:2309.01309
Part 2: in progress