Quantum Bruhat graphs and tilted Richardson varieties

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Outline

- Background: quantum cohomology of the flag variety and two-point curve neighbourhoods.
- Quantum Bruhat graphs and tilted Bruhat intervals.
- Tilted Richardson varieties.
- Tilted Deodhar decomposition and tilted R polynomials.

The Flag Variety

Definition

A complete flag $F_\bullet = (F_0, F_1, \ldots, F_n)$ in $\mathbb{C}^n$ is a nested sequence of vector spaces $\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n$ such that $\dim(F_i) = i$ for $i = 0, 1, \ldots, n$.

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A complete flag $F_\bullet$ can be represented by an $n \times n$ invertible matrix

$$M_F = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \text{ where } F_i = \mathrm{span}\langle v_1, \ldots, v_i \rangle.$$

Two invertible matrices M, M' represent the same complete flag $F_\bullet$ if and only if $M' = Mb$ for some upper-triangular matrix $b \in \mathrm{GL}_n$.

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Proposition

$\mathrm{Fl}_n = \mathrm{GL}_n / B$, where B is the Borel subgroup of all upper-triangular matrices.

Cohomology of Flag Varieties

The cohomology ring $H^*(\mathrm{Fl}_n)$ is a free $\mathbb{Z}$-module generated by the Schubert classes $\{\sigma_w : w \in S_n\}$, where σ_w is the cohomology class of the Schubert variety $X_w := \overline{BwB/B}$.

Schubert Calculus problem

Find a combinatorial interpretation for $c_{u,v}^w$ in the expansion $\sigma_u \cdot \sigma_v = \sum_{w\in S_n} c_{u,v}^w \sigma_w$.

Here $c_{u,v}^w$ counts intersection multiplicity of Schubert varieties $X_u, X_v, X_{w_0 w}$, in general position.

Quantum Cohomology of Flag Varieties

The quantum cohomology ring $QH^*(\mathrm{Fl}_n) \cong H^*(\mathrm{Fl}_n) \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, \ldots, q_{n-1}]$ is a free $\mathbb{Z}[q]$-module generated by the Schubert classes $\{\sigma_w : w \in S_n\}$.

$$\sigma_u \star \sigma_v = \sum_{w \in S_n} \langle \sigma_u, \sigma_v, \sigma_{w_0 w} \rangle_d \, q^d \sigma_w$$

Here $d = (d_1, \ldots, d_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$ and $q^d := q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$.

Similar to $c_{u,v}^w$, the Gromov-Witten invariant $\langle \sigma_u, \sigma_v, \sigma_{w_0 w} \rangle_d$ counts the number of degree d rational curves passing through Schubert varieties $X_u, X_v, X_{w_0 w}$, in general position.

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Motivating Q: What weights q^d appear in the quantum product $\sigma_u \star \sigma_{w_0 v}$? What is the minimal such q^d?
[Fulton-Woodward '04, Postnikov '05, Buch-Chung-Li-Mihalcea '20, Shifler '22]

Curve Neighbourhoods

Besides counting the number of degree d rational curves, it is also natural to study the geometry of the union of all such curves.

Definition (Buch-Chaput-Mihalcea-Perrin '13)

For permutations u, v, the two-point curve neighborhood $\Gamma_d(X^u, X_v)$ is the union of degree d rational curves that passes through both Schubert varieties X^u and X_v in Fl_n.

Fact: $[\Gamma_d(X^u, X_v)] = [q^d]\sigma_u * \sigma_{w_0 v} \in H^*(\mathrm{Fl}_n)$

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Goal: Give a concrete description of $\Gamma_{d_{min}}(X^u, X_v)$ in the minimal degree.

In other words, which flags $F_\bullet \in \Gamma_d(X^u, X_v)$?

Quantum Bruhat Graph

Postnikov ('04) related all quantum degrees appearing in $\sigma_u \star \sigma_{w_0 v}$ to weights of paths on the quantum Bruhat graph.

Definition (Brenti-Fomin-Postnikov '99)

The *quantum Bruhat graph* Γ_n is a weighted directed graph on S_n with the following two types of edges:

$$\begin{cases} w \to wt_{ij} \text{ of weight } 1 & \text{if } \ell(wt_{ij}) = \ell(w) + 1, \\ w \to wt_{ij} \text{ of weight } q_{ij} := q_i \cdots q_{j-1} & \text{if } \ell(wt_{ij}) = \ell(w) + 1 - 2(j-i) \end{cases}$$

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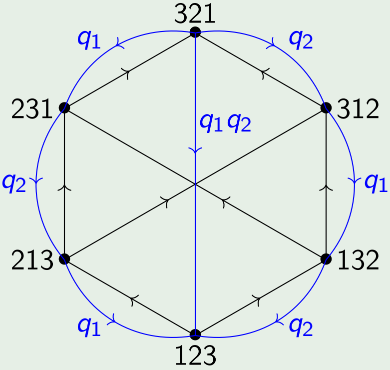
Theorem (Postnikov '04)

There is a unique minimal q^d that appears in $\sigma_u \star \sigma_{w_0 v}$. Such q^d is the weight of any shortest path in quantum Bruhat graph from u to v.

Quantum Bruhat Graph

Example (Quantum Bruhat graph Γ_3)



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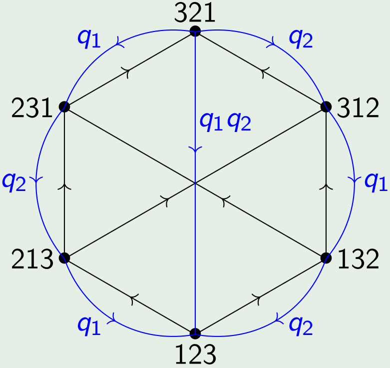

For $u = 231$, $v = 123$, there are two shortest paths, both of which have minimal weight $q^{d_{min}} = q_1 q_2$, which is the minimal q^d that appears in $\sigma_u \star \sigma_{w_0 v}$.

Tilted Bruhat Order

Definition (Brenti-Fomin-Postnikov '99)

For a permutation $w \in S_n$, define the tilted Bruhat order D_w to be the graded partial order "$\leq_w$" on S_n such that $u \leq_w v$ if there is a shortest path from w to v on quantum Bruhat graph passing through u.

Equivalently, let $\ell(u, v)$ be the length of the shortest path on quantum Bruhat graph, then

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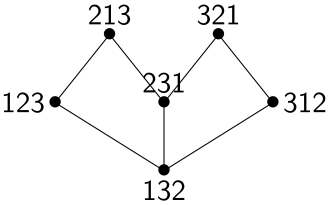

Tilted Bruhat Intervals

For $u \leq_w v$, define the tilted Bruhat interval

$$[u, v]_w := \{x \in S_n : u \leq_w x \leq_w v\}.$$

In particular, $[u, v]_w$ is independent of w as long as $u \leq_w v$. So we will choose $w = u$ and denote the interval as $[u, v]$.

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Theorem (G.-Gao-Gao '23)

The set of T-fixed points (flags represented by permutation matrix) on $\Gamma_{d_{min}}(X^u, X_v)$ is the tilted Bruhat interval $[u, v]$.

Schubert Varieties in terms of rank

Given matrix M, denote $\mathrm{rk}^{SW}_{i,j}(M) :=$ the rank of the $i \times j$ submatrix in the south-west corner of M.

For permutation $v = 3142$, the rank matrix

$$v = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathrm{rk}^{SW}_{i,j}(v) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

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Definition

Let $v \in S_n$ be a permutation. The corresponding Schubert variety/cell is

$$X_v := \left\{ M \in \mathrm{GL_n} \middle| \mathrm{rk}^{SW}_{i,j}(M) \leq \mathrm{rk}^{SW}_{i,j}(v) \right\} / B,$$
$$X_v^\circ := \left\{ M \in \mathrm{GL_n} \middle| \mathrm{rk}^{SW}_{i,j}(M) = \mathrm{rk}^{SW}_{i,j}(v) \right\} / B.$$

If replace south-west (SW) with north-west (NW), we obtain opposite Schubert variety/cell X^u and $(X^u)^\circ$.

Richardson Varieties

For each $u \leq v$ in strong Bruhat order, the Richardson variety/cell are subvarieties of Fl_n, defined as $\mathcal{R}_{u,v} = X^u \cap X_v$ and $\mathcal{R}^\circ_{u,v} = (X^u)^\circ \cap X^\circ_v$.

In terms of rank,

$$\mathcal{R}_{u,v} := \left\{ M \in \mathrm{GL_n} \middle| \begin{array}{c} \mathrm{rk}^{NW}_{i,j}(M) \leq \mathrm{rk}^{NW}_{i,j}(u) \\ \mathrm{rk}^{SW}_{i,j}(M) \leq \mathrm{rk}^{SW}_{i,j}(v) \end{array} \right\} / B.$$

Tilted Richardson Varieties

Definition (G.-Gao-Gao '23)

For $u, v \in S_n$ and a sequence $\mathbf{a} = (a_1, \ldots, a_{n-1})$ such that $u[k] \leq_{a_k} v[k]$, we define the tilted Richardson variety as

$$\mathcal{T}_{u,v,\mathbf{a}} := \left\{ M \in \mathrm{GL_n} \,\middle|\, \begin{matrix} \mathrm{rk}_{i,j}^{\mathbf{a},NW}(M) \leq \mathrm{rk}_{i,j}^{\mathbf{a},NW}(u) \\ \mathrm{rk}_{i,j}^{\mathbf{a},SW}(M) \leq \mathrm{rk}_{i,j}^{\mathbf{a},SW}(v) \end{matrix} \right\} /B.$$

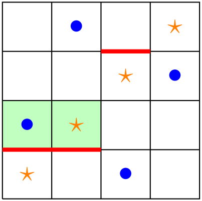
We define the tilted Richardson cell $\mathcal{T}^{\circ}_{u,v,\mathbf{a}}$ by replacing "$\leq$" with "$=$".

Example of Tilted Richardson Varieties

Example

Consider $u = 4321, v = 3142$. We can choose $\mathbf{a} = (4, 4, 2)$.

Example: $j = 2$



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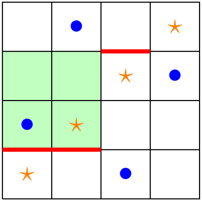

$$\mathrm{rk}^{\mathbf{a},SW}_{2,2}(M) \leq 1$$

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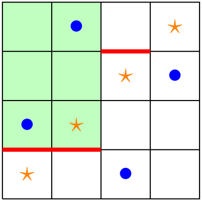

$\mathrm{rk}^{\mathbf{a},SW}_{3,2}(M) \leq 2$

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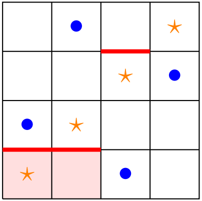

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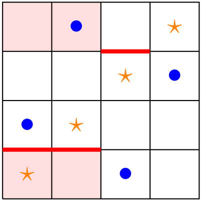

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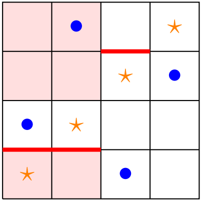

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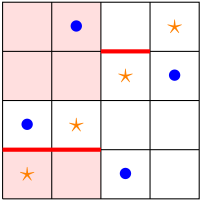

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$\mathcal{T}_{u,v,\mathbf{a}}$ does not depend on the choice of $\mathbf{a}$. So we denote this space as $\mathcal{T}_{u,v}$.

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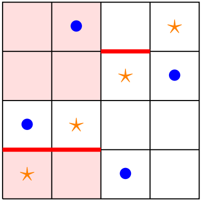

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Example

If $u \leq v$ in strong Bruhat order, and we choose $\mathbf{a} = (1, 1 \ldots, 1)$, we recover the Richardson variety $\mathcal{T}_{u,v} = \mathcal{R}_{u,v}$.

Main Theorem

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$\mathcal{T}_{u,v} = \Gamma_{d_{min}}(X^u, X_v)$.

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The tilted Richardson varieties also have many nice geometric properties:

Theorem (G.-Gao-Gao '23)

- $\mathcal{T}_{u,v}$ *is* T*-invariant and a* T*-fixed point* $x \in \mathcal{T}_{u,v} \iff x \in [u,v]$,
- $\mathcal{T}_{u,v} = \bigsqcup_{[x,y]\subseteq[u,v]} \mathcal{T}^{\circ}_{x,y}$,
- $\mathcal{T}_{u,v} = \overline{\mathcal{T}^{\circ}_{u,v}}$,
- $\dim(\mathcal{T}_{u,v}) = \dim(\mathcal{T}^{\circ}_{u,v}) =$ *length of shortest path from* u *to* v *on* Γ_n,
- $\mathcal{T}_{u,v}$ *is irreducible,*
- $[\mathcal{T}_{u,v}] = [q^{d_{min}}]\sigma_u \star \sigma_{w_0 v} = \sum_w \langle \sigma_u, \sigma_{w_0 v}, \sigma_{w_0 w}\rangle_{d_{min}} \sigma_w \in H^*(\mathrm{Fl}_n)$.

Deodhar decomposition

Motivation: Given a pair of permutations $u \leq v$, the *Kazhdan-Lusztig R polynomial* is defined as

$$R_{u,v}(q) := \#\mathcal{R}^{\circ}_{u,v}(\mathbb{F}_q).$$

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Main idea: In order to understand $R_{u,v}(q)$, Deodhar ('85) introduced Deodhar decomposition, which decomposes $\mathcal{R}^{\circ}_{u,v}$ into simple pieces that are isomorphic to $\mathbb{C}^a \times (\mathbb{C}^*)^b$.

$$\mathcal{R}^{\circ}_{u,v} = \bigsqcup_{\alpha} \mathbb{C}^a \times (\mathbb{C}^*)^b \implies R_{u,v}(q) = \sum_{\alpha} q^a (q-1)^b.$$

Deodhar decomposition

Definition (Marsh-Rietsch '03)

Fix a reduced word $\mathbf{v} = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of v. A *distinguished subword* for u is $\mathbf{u} = u_1 \cdots u_\ell$ where each $u_k \in \{1, s_{i_k}, \color{red}{s_{i_k}}\}$ such that

$$u_k = \begin{cases} 1 \text{ or } s_{i_k}, & \text{if } \ell(u_1 \cdots u_{k-1}) < \ell(u_1 \cdots u_{k-1} s_{i_k}), \\ \color{red}{s_{i_k}}, & \text{if } \ell(u_1 \cdots u_{k-1}) > \ell(u_1 \cdots u_{k-1} s_{i_k}). \end{cases}$$

and their product is u. We denote $\mathbf{u} \prec \mathbf{v}$.

Example

If $\mathbf{v} = s_1 s_2 s_1$, there are two distinguished subwords for $u = \mathrm{id}$:

$$\mathbf{u} = 111 \qquad\qquad \mathbf{u} = s_1 1 \color{red}{s_1}$$

Deodhar decomposition

For any distinguished subword $\mathbf{u} = u_1 \cdots u_\ell \prec \mathbf{v}$, define the Deodhar cell

$$D_{\mathbf{u},\mathbf{v}} := \{g_1 g_2 \cdots g_\ell \cdot B\}/B \subset \mathrm{Fl}_n$$

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$$g_k = \begin{cases} \phi_{i_k}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, p \neq 0, & \text{if } u_k = 1, \\ \phi_{i_k}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{if } u_k = s_{i_k}, \\ \phi_{i_k}\begin{pmatrix} -m & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } u_k = {\color{red} s_{i_k}}. \end{cases}$$

Here $\phi_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & a & b & \\ & & c & d & \\ 0 & & & & \ddots \end{pmatrix}$ embeds into the i^{th} and $(i+1)^{\text{th}}$ row/column of the identity matrix.

Deodhar decomposition

Theorem (Deodhar '85)

$$\mathcal{R}^{\circ}_{u,v} = \bigsqcup_{\mathbf{u} \prec \mathbf{v}} D_{\mathbf{u},\mathbf{v}}, \text{ where each } D_{\mathbf{u},\mathbf{v}} \cong (\mathbb{C}^*)^{\#1\text{'s in } \mathbf{u}} \times \mathbb{C}^{\# s_{i_k} \text{'s in } \mathbf{u}}$$

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$$\mathbf{u} = 111 \qquad\qquad \mathbf{u} = s_1 1 s_1$$

$$D_{111, s_1 s_2 s_1} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ p_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ p_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ p_1 + p_3 & 1 & 0 \\ p_2 p_3 & p_2 & 1 \end{pmatrix} \cong (\mathbb{C}^*)^3.$$

$$D_{s_1 1 s_1, s_1 s_2 s_1} =$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p_2 & 1 \end{pmatrix} \begin{pmatrix} -m_3 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -m_3 & 1 & 0 \\ -p_2 & 0 & 1 \end{pmatrix} \cong \mathbb{C}^* \times \mathbb{C}.$$

Deodhar decomposition (proof idea)

The proof of Deodhar decoposition uses a key recursion:

Proposition (Deodhar '85)

For any simple transposition s_i such that $vs_i < v$, there are canonical isomorphisms

$$\mathcal{R}^{\circ}_{u,v} \cong \begin{cases} \mathcal{R}^{\circ}_{us_i,vs_i}, & \textit{if } us_i < u, \\ \mathcal{R}^{\circ}_{u,vs_i} \times \mathbb{C}^* \sqcup \mathcal{R}^{\circ}_{us_i,vs_i} \times \mathbb{C}, & \textit{if } us_i > u. \end{cases}$$

We can induct on $\ell(u)$ and iteratively construct the decomposition.

Deodhar decomposition (proof idea)

The proof of Deodhar decoposition uses a key recursion:

Proposition (G.-Gao-Gao '23)

For any simple transposition s_i *such that there exists sequence* $\mathbf{a}$ *where* $u \leq_{\mathbf{a}} v$, $vs_i <_{\mathbf{a}} v$, *and* $a_{i-1} = a_i = a_{i+1}$, *there are canonical isomorphisms*

$$\mathcal{T}^{\circ}_{u,v} \cong \begin{cases} \mathcal{T}^{\circ}_{us_i,vs_i}, & \text{if } us_i <_{\mathbf{a}} u, \\ \mathcal{T}^{\circ}_{u,vs_i} \times \mathbb{C}^* \sqcup \mathcal{T}^{\circ}_{us_i,vs_i} \times \mathbb{C}, & \text{if } us_i >_{\mathbf{a}} u. \end{cases}$$

The tilted version serves as a building block for a Deodhar-like decomposition for $\mathcal{T}^{\circ}_{u,v}$.

Tilted Deodhar decomposition

Definition (G.-Gao-Gao '23)

Fix a *tilted reduced word* $\mathbf{v} = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of v. A *tilted distinguished subword* for u is $\mathbf{u} = u_1 \cdots u_\ell$ where each $u_k \in \{1, s_{i_k}, s_{i_k}, s_{i_k}\}$ that follows certain rules and their product is u. We denote $\mathbf{u} \prec_t \mathbf{v}$.

Example

For $u = 512346$ and $v = 246513$, given a tilted reduced word $\mathbf{v} = s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_3 s_2 s_1 s_1 s_2$, there are 4 tilted distinguished subwords:

$\mathbf{u} = 111111 s_4 s_3 s_2 s_1 11$

$\mathbf{u} = 111 s_1 11 s_4 s_3 s_2 s_1 1 s_2$

$\mathbf{u} = s_3 1111 s_3 s_4 s_3 s_2 s_1 11$

$\mathbf{u} = s_3 11 s_1 1 s_3 s_4 s_3 s_2 s_1 1 s_2$

Tilted Deodhar decomposition

For any $\mathbf{u} = u_1 \cdots u_\ell \prec_t \mathbf{v}$, define the tilted Deodhar cell

$$D_{\mathbf{u},\mathbf{v}} := \{g_1 g_2 \cdots g_\ell \cdot B\} / B \subset \mathrm{Fl}_n$$

where

$$g_k = \begin{cases} \phi_{i_k}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, p \neq 0, & \text{if } u_k = 1, \\ \phi_{i_k}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{if } u_k = s_{i_k}, \end{cases} \begin{cases} \phi_{i_k}\begin{pmatrix} -m & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } u_k = {\color{red}s_{i_k}}, \\ \phi_{i_k}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \text{if } u_k = {\color{blue}s_{i_k}}. \end{cases}$$

Theorem (G.-Gao-Gao '23)

$$\mathcal{T}^\circ_{u,v} = \bigsqcup_{\mathbf{u} \prec_t \mathbf{v}} D_{\mathbf{u},\mathbf{v}}, \ \textit{where each } D_{\mathbf{u},\mathbf{v}} \cong (\mathbb{C}^*)^{\#1\text{'s in } \mathbf{u}} \times \mathbb{C}^{\#{\color{red}s_{i_k}}\text{'s in } \mathbf{u}}$$

Why tilted Deodhar?

Reason 1: Proof of irreducibility of $\mathcal{T}_{u,v}$

Theorem (G.-Gao-Gao '23)

For any pair of permutations u, v, given a tilted reduced word $\mathbf{v}$, there exists a unique subword $\mathbf{u}_0 \prec_t \mathbf{v}$ that does not use s_{i_k}, called the positive distinguished subword. The corresponding tilted Deodhar cell $D_{\mathbf{u}_0,\mathbf{v}} \cong (\mathbb{C}^*)^{\ell(u,v)}$ is the unique cell of maximal dimension, and all other cells $D_{\mathbf{u},\mathbf{v}}$ have dimension $< \ell(u, v)$.

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Corollary (G.-Gao-Gao '23)

$\mathcal{T}_{u,v}$ *is irreducible, and* $D_{\mathbf{u}_0,\mathbf{v}} \subset \mathcal{T}_{u,v}$ *is a dense subset.*

Why tilted Deodhar?

Reason 2: Total Positivity

Theorem (Björner '84, Dyer '93)

If a poset is thin and EL-shellable, then the poset is the face poset of a regular CW-complex. In particular, the strong Bruhat order is thin and EL-shellable.

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Motivating Q: Can one find a natural realization of such CW-complex?

Answer: Totally Nonnegative Flag Variety $\mathrm{Fl}_n^{\geq 0} = \overline{\mathrm{Fl}_n^{>0}}$, where $\mathrm{Fl}_n^{>0}$ contain complete flags represented by TNN matrices (matrices with nonnegative minors).

Why tilted Deodhar?

Theorem (Marsh-Rietsch '03)

$$D_{\mathbf{u},\mathbf{v}} \cap \mathrm{Fl}_n^{\geq 0} = \begin{cases} \mathbb{R}_{>0}^{\ell(v)-\ell(u)}, & \textit{if } \mathbf{u} = \mathbf{u}_0 \textit{ unique positive subword,} \\ \emptyset, & \textit{otherwise.} \end{cases}$$

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Corollary (Rietsch-Williams '08)

$\mathcal{R}_{u,v}^{\circ} \cap \mathrm{Fl}_n^{\geq 0} \cong \mathbb{R}_{>0}^{\ell(v)-\ell(u)}$, *in particular they form a CW-complex.*

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Theorem (Galashin-Karp-Lam '21)

This is a regular CW-complex.

Q: Since Brenti-Fomin-Postnikov '99 proved that tilted Bruhat intervals are thin and EL-shellable, can we develop a similar story using $\mathcal{T}_{u,v}$ and tilted Deodhar decomposition?

Tilted R polynomial

Given a pair of permutations, the tilted R polynomial is defined as

$$R^{\mathrm{tilt}}_{u,v}(q) := \#\mathcal{T}^{\circ}_{u,v}(\mathbb{F}_q).$$

We can use tilted Deodhar decomposition to compute these polynomials.

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Combinatorial Invariance Problem (Lusztig '83, Dyer '87)

The Kazhdan-Lusztig polynomial $P_{u,v}(q)$ depends only on the poset structure of $[u, v]$.

This can be implied by the combinatorial invariance of R polynomials.

Conjecture (G.-Gao-Gao '23)

$R^{tilt}_{u,v}(q)$ depends only on the poset structure of tilted Bruhat interval $[u, v]$.

Thank you all for listening!