Quantum Bruhat graphs & Tilted Richardson Varieties.

Brandeis Comb. seminar 10/10

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0. Motivation

Let Fl_n be the complete flag variety

$$Fl_n = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\} = GL_n(\mathbb{C})/\underset{\substack{\downarrow \\ \text{upper triangular matrix}}}{B}$$

$\dim(V_i) = i$

Schubert calculus $\Rightarrow$ $H^*(Fl_n)$ is a free $\mathbb{Z}$-module generated by Schubert classes

$\{[X_w] : w \in S_n\}$

Major open Q: $[X_u]\cdot[X_v] = \sum_{w\in S_n} c_{u,v}^{w}[X_w]$ LR-coeffs

count intersection mult. of Schub. varieties in general position

The Quantum cohomology ring

$$QH^*(Fl_n) \cong H^*(Fl_n) \otimes_{\mathbb{Z}} \underbrace{\mathbb{Z}[q_1, q_2, \cdots, q_{n-1}]}_{\mathbb{Z}[q]}$$

is a free $\mathbb{Z}[q]$-module with basis $\{[X_w] : w \in S_n\}$

$$[X_u] * [X_v] = \sum_{\substack{w \in S_n \\ d=(d_1, d_2, \cdots, d_{n-1})}} \underbrace{\langle X_u, X_v, X_{w_0 w} \rangle_d}_{\text{GW-invariant}} q^d [X_w]$$

GW-invariant count # of deg d rational curves passing through Schub. varieties in general position

<ins>Rmk</ins>: If $d=0$. curve = point $\Rightarrow$ GW-invariant $= C_{uv}^w$

<ins>Motivating</ins> Q: What are the degrees d that appear in $[X_u] * [X_v]$?

Fulton-Woodward '04 Postnikov '05

Buch-Chung-Li-Mihalcea '20 Schiffler '22

<ins>Today</ins>: Focus on minimal q-degree d_{min}

- Simple formula for computing d_{min}
- degree d_{min} part of quantum product

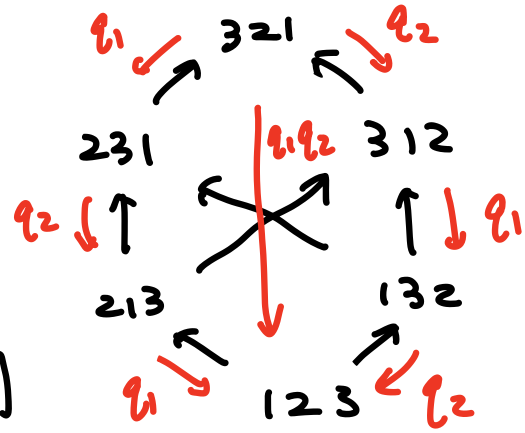
1. Quantum Bruhat Graph

- A weighted directed graph on S_n

  (Defined by Brenti - Fomin - Postnikov '99)

- Edges $\begin{cases} w \xrightarrow{1} w\cdot t_{ij} & \text{if } \ell(wt_{ij}) = \ell(w)+1 \\ w \xrightarrow{q_{ij}} w\cdot t_{ij} & \text{if } \ell(wt_{ij}) = \ell(w) - \ell(t_{ij}) \end{cases}$

  $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$ $\qquad\qquad = \ell(w) - 2(j-i)+1$



Thm: (Postnikov '05)

All shortest path from $u \to v$ have the same weight, and this weight is $q^{d_{\min}}$ in $[X_u] * [X_{w_0 v}]$

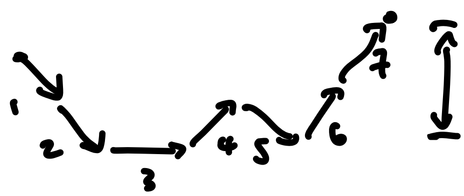
A simpler formula for $q^{d_{\min}} = q_1^{d_1} q_2^{d_2} \cdots q_{n-1}^{d_{n-1}}$

Given $u, v \in S_n$ $\quad u[k] := \{u_1, u_2, \cdots, u_k\}$

e.g. $u = 4637521 \quad v = 5312467$, $k=4$

$u[4] = \{3,4,6,7\} \quad v[4] = \{1,2,3,5\}$

Lattice Path



depth $(u[4], v[4])$

$\begin{cases} \nearrow & \text{if } i \in u[k] \setminus v[k] \\ \searrow & \text{if } i \in v[k] \setminus u[k] \\ \rightarrow & \text{else} \end{cases}$

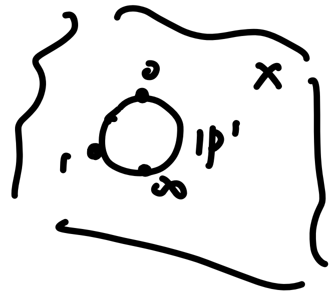
Thm: (G-Gao-Gao ; BCLM'20)

$$q^{\dim} = \prod_{k=1}^{n-1} q_k^{\operatorname{depth}(u[k],\, v[k])}$$

2. Tilted Bruhat order

$M_d = \widetilde{M}_{0,3}(X, d)$ = space of degree d genus 0 rational curves on X w/ 3 marked pts

$ev_i : M_d \to X \quad i = 1, 2, 3.$



- The GW-variety

  $M_d(X_w, X^v) = ev_1^{-1}(X_w) \cap ev_2^{-1}(X^v)$

- Two-pointed curve neighbourhood

  $\Gamma_d(X_u, X^v) = ev_3(M_d(X_u, X^v))$

  = union of deg. d curves passing through X_u, X^v (subvariety of X)

- $(ev_3)_*[M_d(X_u, X^v)]$ = deg d component in $[X_u] * [X^v]$

$$(ev_3)_*[M_d(X_u, X^v)] = \sum_{w \in S_n} \langle X_u, X_{w_0 v}, X_{w_0 w} \rangle_d [X_w]$$

$$(ev_3)_*[M_d(X_u, X^v)] = c \cdot [\Gamma_d(X_u, X^v)], \quad c \in \mathbb{Z}_{\geq 0}$$

Goal: Study these curve neighbourhoods $\Gamma_d(X_u, X^v)$

Li-M '14   KLS '13

Buch-M '15   BCMP '17

- $X = Gr(k,n)$ KLS'13   Positroid varieties

- $X = Fl_n$
  - $d=0$ $\Gamma_d(X_u, X^v) = X_u \cap X^v = X_u^v$ Richardson varieties

  - $d = (0, 0 \cdots, 1, 0 \cdots 0)$: also Richardson

- L-M suspected that all Γ_d are Richardsons

  Not quite, but close

We focus on $\Gamma_{d_{min}}(X_u, X^v)$ in $d = d_{min}$

Just like $X_u^v \longleftrightarrow$ strong Bruhat order

$$\Gamma_{d_{min}}(X_u, X^v) \longleftrightarrow \text{tilted Bruhat order}$$

<u>Def</u>: [Brenti-Fomin-Pos '99]

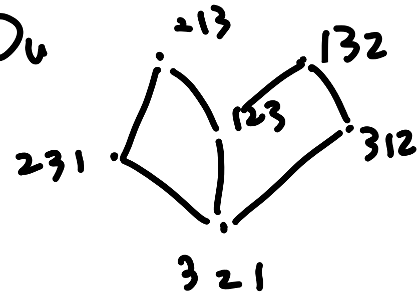
Fix $u \in S_n$, the tilted Bruhat order D_u is defined as

$w \leq_u v$ if there is a shortest path (in QBG) from

$u \to v$ that passes through w

i.e. $\ell(u,v) = \ell(u,w) + \ell(w,v)$

<u>e.g.</u> $u = 321$ D_u



Def: For $X, Y \subset [n] = \{1,2,\cdots,n\}$, $|X| = |Y|$

we say that $X \leq Y$ in Gale order, if after sorting

$X = \{x_1 \leq x_2 \leq \cdots \leq x_k\}$, $Y = \{y_1 \leq y_2 \leq \cdots \leq y_k\}$

we have $x_i \leq y_i$ for all $i = 1, 2, \cdots, k$

We say that $X \leq_a Y$ in shifted Gale order

if $(X-a) \% n \leq (Y-a) \% n$

i.e. sort X, Y as if $a < a+1 < a+2 < \cdots < n < 1 < \cdots < a-1$

Fact: For any $X, Y \subset [n]$, $|X| = |Y|$, there exists a s.t.

$X \leq_a Y$

Thm: (G-Gao-Gao)

$w \underset{u}{\leq} v$ in tilted Bruhat order D_u

$\Leftrightarrow$ for any/all sequence $\underline{a} = (a_1, a_2, \cdots, a_{n-1})$ s.t.

$u[k] \leq_{a_k} v[k]$, we have $u[k] \leq_{a_k} w[k] \leq_{a_k} v[k]$

Ex: $123 \underset{132}{\leq} 213$ $\quad\quad$ $\underline{a} = (1,2)$ or $(3,2)$

$\{1\} \leq_{1,3} \{2\}$ $\quad\quad$ $\{1,2\} \leq_2 \{1,3\}$

$\{1\}$ $\quad\quad$ $\{1,2\}$

Def: Tilted Bruhat interval. $[u,v]_w := \{x \in S_n : u \leq_w x \leq_w v\}$

interval in D_w.

Fact: $[u,v]_w$ is independent of w, so we can choose $w=u$ and denote the interval as $[u,v]$

3. Tilted Richardson variety

Goal: Give a "geometric realization" of tilted Bruhat order $[u,v]$

A complete flag $V_\bullet \in Fl_n$ can be represented by matrix
$$V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$$

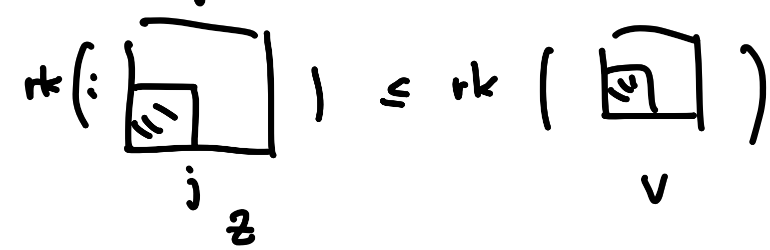
$$Z = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \qquad V_i = \operatorname{span}(v_1, v_2, \ldots v_i)$$

Given $v = 3142$, $rk^{sw}_{i,j}(v) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ rank of SW submatrix

$$\begin{bmatrix} & 1 & & \\ & & & 1 \\ 1 & & & \\ & & 1 & \end{bmatrix}$$

Schubert variety

$$X_v = \{ Z \in GL_n \mid rk^{sw}_{i,j}(Z) \le rk^{sw}_{i,j}(v) \} / B$$



Similarly, $u = 2143$ $\quad rk^{NW}_{i,j}(u) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ rank of NW submatrix

$$\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$$

$$X^u = \{ z \in GL_n \mid rk^{NW}_{i,j}(z) \le rk^{NW}_{i,j}(u) \} / B$$

Richardson variety $R_{u,v} = X^u \cap X_v$

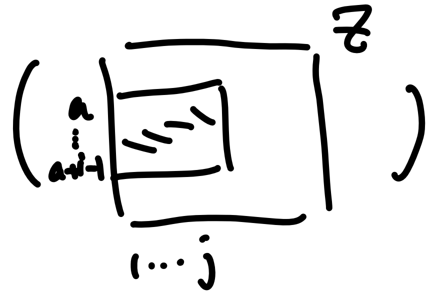
$$= \{ z \in GL_n \mid rk^{NW}_{i,j}(z) \le rk^{NW}_{i,j}(u),\quad rk^{SW}_{i,j}(z) \le rk^{SW}_{i,j}(v) \} / B$$

__Def:__ Given $u, v \in S_n$ and a sequence $\underline{a} = (a_1, a_2, \dots, a_n)$ s.t. $u[k] \le_{a_k} v[k]$ for all k, define the __tilted Richardson variety__

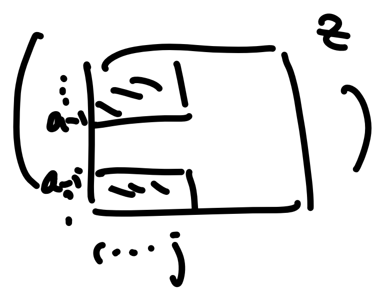
$$T_{u,v,\underline{a}} = \{ z \in GL_n \mid rk^{NW,a_j}_{i,j}(z) \le rk^{NW,a_j}_{i,j}(u),\quad rk^{SW,a_j}_{i,j}(z) \le rk^{SW,a_j}_{i,j}(v) \} / B$$

$\cap$
Fl_n

where $rk^{NW,a}_{i,j}(z) =$ rank

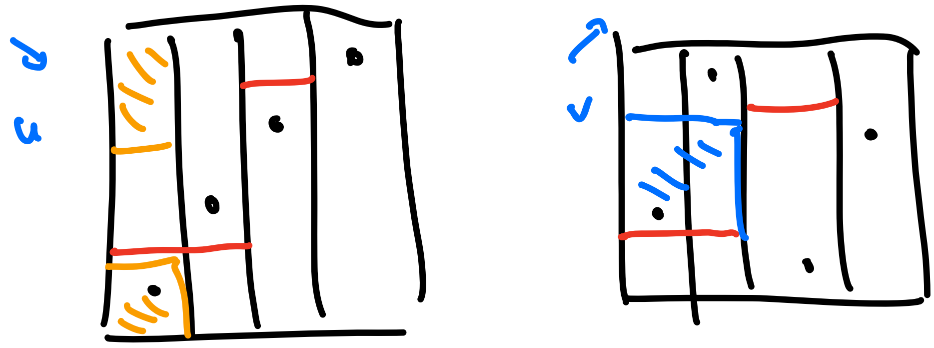


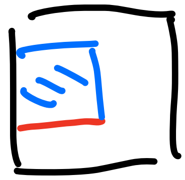
$rk^{SW,a}_{i,j}(z) =$ rank

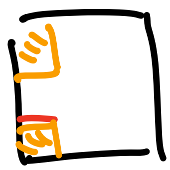


Define $T^{\circ}_{u,v,\underline{a}}$ by replacing "$\le$" with "$=$"

Ex: $u = 4321$, $v = 3142$, $\underline{a} = (4,4,2)$



$T_{u,v,\underline{a}} = \{ z \in GL_n \mid$ rk() ≤ 1

rk() ≤ 1

and other few rank conditions $\} / B$

Rmk: If $u \leq v$, $\underline{a} = (1,1,\cdots,1)$

$$T_{u,v,\underline{a}} = R_{u,v}$$

Main Thm (G-Gao-Gao)

① $T_{u,v,\underline{a}}$, $T^\circ_{u,v,\underline{a}}$ are indep. of $\underline{a}$ (as long as $u[k] \leq_{a_k} v[k]$)

② $T^\circ_{u,v} \neq \emptyset$, $T_{u,v} \neq \emptyset$

③ $T_{u,v} = \coprod_{[x,y] \leq [u,v]} T^\circ_{x,y}$

(4) $T_{u,v} = \overline{T^{\circ}_{u,v}}$ is irreducible

(5) $\dim(T_{u,v}) = \dim(T^{\circ}_{u,v}) = \ell(u,v)$

(6) $[X_u] * [X_{w_0 v}] = q^{d_{min}} \underset{\substack{\cap \\ H^*(Fl_n)}}{[T_{u,v}]}$ + higher order terms