The quantum Bruhat graph and tilted Richardson varieties

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Outline

- Quantum Bruhat graphs and the lowest quantum degree in product of Schubert classes in $QH^*(\mathrm{Fl}_n)$.
- Tilted Bruhat orders and an analogue of Ehresmann criterion.
- Tilted Richardson varieties: a concrete description of two-point curve neighbourhoods in terms of tilted Bruhat orders.

The Flag Variety

Definition

A complete flag $F_\bullet = (F_0, F_1, \ldots, F_n)$ in $\mathbb{C}^n$ is a nested sequence of vector spaces $\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n$ such that $\dim(F_i) = i$ for $i = 0, 1, \ldots, n$.

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A complete flag $F_\bullet$ can be represented by an $n \times n$ invertible matrix

$$M_F = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \text{ where } F_i = \mathrm{span}\langle v_1, \ldots, v_i \rangle.$$

Two invertible matrices M, M' represent the same complete flag $F_\bullet$ if and only if $M' = Mb$ for some upper-triangular matrix $b \in \mathrm{GL}_n$.

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Proposition

$\mathrm{Fl}_n = \mathrm{GL}_n / B$, where B is the Borel subgroup of all upper-triangular matrices.

Cohomology of Flag Varieties

The cohomology ring $H^*(\mathrm{Fl}_n)$ is a free $\mathbb{Z}$-module generated by the Schubert classes $\{\sigma_w : w \in S_n\}$, where σ_w is the cohomology class of the Schubert variety $X_w := \overline{BwB/B}$.

Schubert Calculus problem

Find a combinatorial interpretation for $c^w_{u,v}$ in the expansion $\sigma_u \cdot \sigma_v = \sum_{w \in S_n} c^w_{u,v} \sigma_w$.

Here $c^w_{u,v}$ counts intersection multiplicity of Schubert varieties $X_u, X_v, X_{w_0 w}$, in general position.

Quantum Cohomology of Flag Varieties

The quantum cohomology ring $QH^*(\mathrm{Fl}_n) \cong H^*(\mathrm{Fl}_n) \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, \ldots, q_{n-1}]$ is a free $\mathbb{Z}[q]$-module generated by the Schubert classes $\{\sigma_w : w \in S_n\}$.

$$\sigma_u \star \sigma_v = \sum_{w \in S_n} \langle \sigma_u, \sigma_v, \sigma_{w_0 w} \rangle_d \, q^d \sigma_w$$

Here $d = (d_1, \ldots, d_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$ and $q^d := q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$.

Similar to $c_{u,v}^w$, the Gromov-Witten invariant $\langle \sigma_u, \sigma_v, \sigma_{w_0 w} \rangle_d$ counts the number of degree d rational curves passing through Schubert varieties $X_u, X_v, X_{w_0 w}$, in general position.

Motivating Q: What weights q^d appear in the quantum product $\sigma_u \star \sigma_{w_0 v}$? What is the minimal such q^d?
[Fulton-Woodward '04, Postnikov '04, Buch-Chung-Li-Mihalcea '20, Shifler '22]

Quantum Bruhat Graph

Postnikov ('04) related all quantum degrees appearing in $\sigma_u \star \sigma_{w_0 v}$ to weights of paths on the quantum Bruhat graph.

Definition (Brenti-Fomin-Postnikov '99)

The *quantum Bruhat graph* Γ_n is a weighted directed graph on S_n with the following two types of edges:

$$\begin{cases} w \to wt_{ij} \text{ of weight } 1 & \text{if } \ell(wt_{ij}) = \ell(w) + 1, \\ w \to wt_{ij} \text{ of weight } q_{ij} := q_i \cdots q_{j-1} & \text{if } \ell(wt_{ij}) = \ell(w) + 1 - 2(j-i) \end{cases}$$

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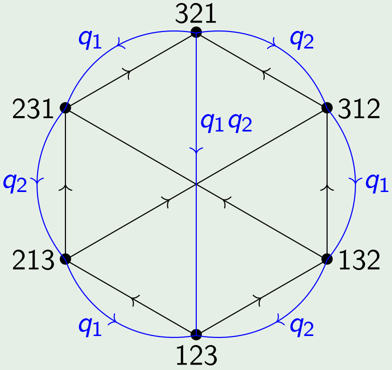
Theorem (Postnikov '04)

There is a unique minimal q^d that appears in $\sigma_u \star \sigma_{w_0 v}$. Such q^d is the weight of any shortest path in quantum Bruhat graph from u to v.

Quantum Bruhat Graph

Example (Quantum Bruhat graph Γ_3)



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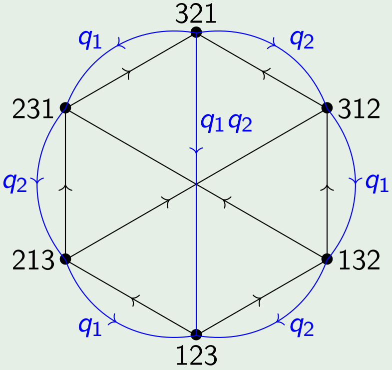


For $u = 231$, $v = 123$, there are two shortest paths, both of which have minimal weight $q^{d_{min}} = q_1 q_2$, which is the minimal q^d that appears in $\sigma_u \star \sigma_{w_0 v}$.

A Simple Formula for $q^{d_{min}}$

Theorem (G.-Gao-Gao '23, Buch-Chung-Li-Mihalcea '20)

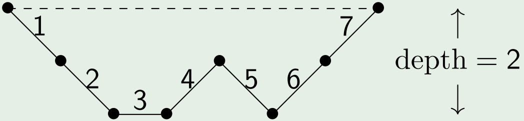
For $u, v \in S_n$, the minimal quantum degree $q^{d_{min}}$ in $\sigma_u \star \sigma_{w_0 v}$ is

$$q^{d_{min}} = \prod_{k=1}^{n-1} q_i^{\mathrm{depth}(u[k], v[k])}.$$

Here $u[k] := \{u(1), \ldots, u(k)\}$.

Example

For $u = 4637521, v = 5312467$ and $k = 4$, $u[k] = \{3, 4, 6, 7\}$ and $v[k] = \{1, 2, 3, 5\}$. We have $\mathrm{depth}(u[k], v[k]) = 2$.



Total $q^{d_{min}} = q_2 q_3 q_4^2 q_5^2 q_6$.

Curve Neighbourhoods

Besides counting the number of degree d rational curves, it is also natural to study the geometry of the union of all such curves.

Definition (Buch-Chaput-Mihalcea-Perrin '13)

For permutations u, v, the two-point curve neighborhood $\Gamma_d(X^u, X_v)$ is the union of degree d rational curves that passes through both Schubert varieties X^u and X_v in Fl_n.

Fact: $[\Gamma_d(X^u, X_v)] = \sum_{w \in S_n} \langle \sigma_u, \sigma_{w_0 v}, \sigma_{w_0 w} \rangle_d \, \sigma_w \in H^*(\mathrm{Fl}_n)$

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Goal: Give a concrete description of $\Gamma_{d_{min}}(X^u, X_v)$ in the minimal degree. In other words, which flags $F_\bullet \in \Gamma_d(X^u, X_v)$?

Tilted Bruhat Order

Definition (Brenti-Fomin-Postnikov '99)

For a permutation $w \in S_n$, define the tilted Bruhat order D_w to be the graded partial order "$\leq_w$" on S_n such that $u \leq_w v$ if there is a shortest path from w to v on quantum Bruhat graph passing through u.

Equivalently, let $\ell(u, v)$ be the length of the shortest path on quantum Bruhat graph, then

$$u \leq_w v \iff \ell(w, u) + \ell(u, v) = \ell(w, v).$$

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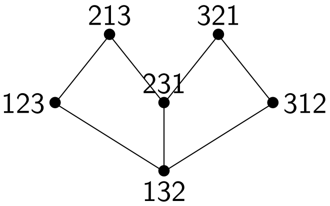

Ehresmann Criterion for Strong Bruhat Order

For $X, Y \subset [n] := \{1, 2, \ldots, n\}$, $|X| = |Y| = k$, we say $X \leq Y$ in Gale order, if after sorting $X = \{x_1 < \cdots < x_k\}$, $Y = \{y_1 < \cdots < y_k\}$, we have

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Theorem (Ehresmann 1934)

For permutations $u, v \in S_n$, $u \leq v$ in strong Bruhat order if and only if

$$u[k] := \{u(1), \ldots, u(k)\} \leq \{v(1), \ldots, v(k)\} =: v[k]$$

for all $k \in [n-1]$.

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Example

Let $u = 1324, v = 2413$, $u \le v$. Indeed we have

$$\{1\} \le \{2\}, \{1, 3\} \le \{2, 4\} \text{ and } \{1, 2, 3\} \le \{1, 2, 4\}.$$

An Ehresmann-like Criterion for Tilted Bruhat Order

For $a \in [n]$, let $\leq_a$ be the shifted order of $[n]$ where

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Fun exercise: For any $X, Y \subset [n]$, $|X| = |Y| = k$, there exists an $a \in [n]$ such that $X \leq_a Y$.

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Theorem (G.-Gao-Gao '23)

$u \leq_w v$ iff for any/all sequence $\mathbf{a} = (a_1, \ldots, a_{n-1})$ such that $w[k] \leq_{a_k} v[k]$, we have

$$w[k] \leq_{a_k} u[k] \leq_{a_k} v[k].$$

Example of Tilted Bruhat Order

Let $w = 132, u = 123, v = 213$.

On quantum Bruhat graph, $w \to u \to v$ is a shortest path from w to v. Hence $u \leq_w v$.

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Using our criterion, $w[k] \leq_{a_k} v[k]$ for $\mathbf{a} = (a_1, a_2) = (1, 2)$. Indeed, $w[k] \leq_{a_k} u[k] \leq_{a_k} v[k]$ since

$$\{1\} \leq_1 \{1\} \leq_1 \{2\}$$
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Alternatively, $w[k] \leq_{a_k} v[k]$ also for $\mathbf{a} = (a_1, a_2) = (3, 2)$. Indeed,

$$\{1\} \leq_3 \{1\} \leq_3 \{2\}$$
$$\{1, 3\} \leq_2 \{1, 2\} \leq_2 \{1, 2\}.$$

Tilted Bruhat Intervals

For $u \leq_w v$, define the tilted Bruhat interval

$$[u, v]_w := \{x \in S_n : u \leq_w x \leq_w v\}.$$

In particular, $[u, v]_w$ is independent of w as long as $u \leq_w v$. So we will choose $w = u$ and denote the interval as $[u, v]$.

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Theorem (G.-Gao-Gao '23)

The set of T-fixed points (flags represented by permutation matrix) on $\Gamma_{d_{min}}(X^u, X_v)$ is the tilted Bruhat interval $[u, v]$.

Schubert Varieties in terms of rank

Given matrix M, denote $\mathrm{rk}_{i,j}^{SW}(M) :=$ the rank of the $i \times j$ submatrix in the south-west corner of M.
For permutation $v = 3142$, the rank matrix

$$v = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathrm{rk}_{i,j}^{SW}(v) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

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Definition

Let $v \in S_n$ be a permutation. The corresponding Schubert variety/cell is

$$X_v := \left\{ M \in \mathrm{GL_n} \middle| \mathrm{rk}^{SW}_{i,j}(M) \leq \mathrm{rk}^{SW}_{i,j}(v) \right\} / B,$$
$$X_v^\circ := \left\{ M \in \mathrm{GL_n} \middle| \mathrm{rk}^{SW}_{i,j}(M) = \mathrm{rk}^{SW}_{i,j}(v) \right\} / B.$$

If replace south-west (SW) with north-west (NW), we obtain opposite Schubert variety/cell X^u and $(X^u)^\circ$.

Richardson Varieties

For each $u \leq v$ in strong Bruhat order, the Richardson variety/cell are subvarieties of Fl_n, defined as $\mathcal{R}_{u,v} = X^u \cap X_v$ and $\mathcal{R}^\circ_{u,v} = (X^u)^\circ \cap X^\circ_v$.

In terms of rank,

$$\mathcal{R}_{u,v} := \left\{ M \in \mathrm{GL_n} \,\middle|\, \begin{matrix} \mathrm{rk}^{NW}_{i,j}(M) \leq \mathrm{rk}^{NW}_{i,j}(u) \\ \mathrm{rk}^{SW}_{i,j}(M) \leq \mathrm{rk}^{SW}_{i,j}(v) \end{matrix} \right\} / B.$$

Tilted Richardson Varieties

Definition (G.-Gao-Gao '23)

For $u, v \in S_n$ and a sequence $\mathbf{a} = (a_1, \dots, a_{n-1})$ such that $u[k] \leq_{a_k} v[k]$, we define the tilted Richardson variety as

$$\mathcal{T}_{u,v,\mathbf{a}} := \left\{ M \in \mathrm{GL_n} \middle| \begin{array}{l} \mathrm{rk}_{i,j}^{a_j,NW}(M) \leq \mathrm{rk}_{i,j}^{a_j,NW}(u) \\ \mathrm{rk}_{i,j}^{a_j,SW}(M) \leq \mathrm{rk}_{i,j}^{a_j,SW}(v) \end{array} \right\} / B.$$

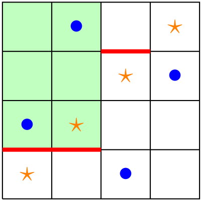
Here $\mathrm{rk}_{i,j}^{a,NW}(M) :=$ the rank of the submatrix of M in row index $a, a+1, \dots a+i-1$ (modulo n) and column index $1, \dots, j$;
$\mathrm{rk}_{i,j}^{a,SW}(M) :=$ the rank of the submatrix of M in row index $a-1, a-2, \dots a-i$ (modulo n) and column index $1, \dots, j$.

We define the tilted Richardson cell $\mathcal{T}^{\circ}_{u,v,\mathbf{a}}$ by replacing "$\leq$" with "$=$".

Example of Tilted Richardson Varieties

Example

Consider $u = 4321, v = 3142$. We can choose $\mathbf{a} = (4, 4, 2)$.



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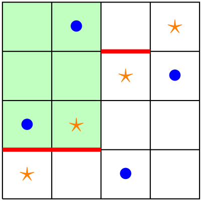

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Theorem (G.-Gao-Gao '23)

$\mathcal{T}_{u,v,\mathbf{a}}$ does not depend on the choice of $\mathbf{a}$. So we denote this space as $\mathcal{T}_{u,v}$.

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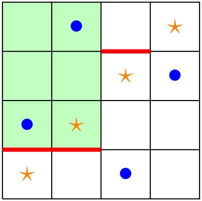

$\mathrm{rk}_{3,2}^{4,SW}(M) \leq 2$

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Example

If $u \leq v$ in strong Bruhat order, and we choose $\mathbf{a} = (1, 1 \ldots, 1)$, we recover the Richardson variety $\mathcal{T}_{u,v} = \mathcal{R}_{u,v}$.

Main Theorem

Theorem (G.-Gao-Gao '23)

$\mathcal{T}_{u,v} = \Gamma_{d_{min}}(X^u, X_v)$.

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The tilted Richardson varieties also have many nice geometric properties:

Theorem (G.-Gao-Gao '23)

- $\mathcal{T}_{u,v}$ *is* T*-invariant and a* T*-fixed point* $x \in \mathcal{T}_{u,v} \iff x \in [u, v]$.
- $\mathcal{T}_{u,v} = \bigsqcup_{[x,y]\subseteq[u,v]} \mathcal{T}^{\circ}_{x,y}$,
- $\mathcal{T}_{u,v} = \overline{\mathcal{T}^{\circ}_{u,v}}$,
- $\dim(\mathcal{T}_{u,v}) = \dim(\mathcal{T}^{\circ}_{u,v}) =$ *length of shortest path from* u *to* v *on* Γ_n,
- $\mathcal{T}_{u,v}$ *is irreducible,*
- $[\mathcal{T}_{u,v}] = [q^{d_{min}}]\sigma_u \star \sigma_{w_0 v} = \sum_w \langle \sigma_u, \sigma_{w_0 v}, \sigma_{w_0 w}\rangle_{d_{min}} \sigma_w \in H^*(\mathrm{Fl}_n)$.

Thank you all for listening!