Trivial Notions

Riemann-Roch on Graphs

Outline:

1. Divisor Theory on Graphs
2. Riemann-Roch on Graphs
3. Geometric picture
4. An application

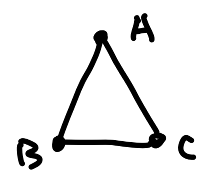
I. Divisor theory on Graphs

Start with a connected graph $G = (V, E)$ (allow multi-edges, no self-loops for now)

Def: A divisor D on G is a formal $\mathbb{Z}$-sum of vertices

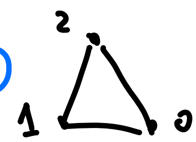
$$D = \sum_{v \in V} D(v) \cdot v$$

Ex:



$D = A + 2B$

(negative chips mean in debt)



"poker chip on vertices"

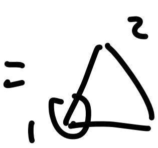
Def: $\deg D = \sum_{v \in V} D(v)$

D is effective if $\forall v \in V,\ D(v) \geq 0$ $\quad (D \geq 0)$

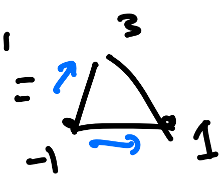
$\deg(D) = 3$

D is effective

Ex: Fire at B.

$D =$ 

$\downarrow$

$D' =$ 

What about linear equivalence?

Play "chip-firing game"

Def: A chip-firing move at vertex v takes $D \to D'$ where

$$D'(u) = \begin{cases} D(u) + \#(u - v) & \text{if } u \neq v \\ D(v) - \operatorname{val}(v) & \text{if } u = v \end{cases}$$

Two divisors $D_1 \sim D_2$ are linearly equivalent if we can send D_1 to D_2 by a sequence of chip-firing moves

$$\mathrm{Pic}(G) = \mathrm{Div}(G)/\sim$$

Ex 1: $\mathrm{Pic}^0(\Delta) \cong \mathbb{Z}/3\mathbb{Z}$

$$\mathrm{Pic}^0(G) = \mathrm{Div}^0(G)/\sim$$

Ex 2: $|\mathrm{Pic}^0(G)| =$ # spanning trees in G

$\downarrow$ ($\mathrm{Jac}(G)$)

Jacobian / sandpile group / critical group

finite abelian group.

II. RR on Graphs

Thm: (Riemann-Roch) For any graph G and $D \in \mathrm{Div}(G)$

$$rk(D) - rk(K-D) = \deg D - g + 1$$

[Baker-Norine '07]

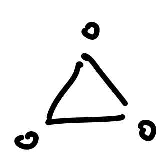
genus g = topological genus of G viewed as a 1-complex

$= E - V + 1$

Ex: genus $\Delta = 1$

Canonical Divisor K: $K(v) = \mathrm{val}(v) - 2 \quad \forall v \in V$

$$\deg K = 2g-2$$

Ex: $K =$ 

rank of divisor D?

Recall on a curve C, $rk(D) = \ell(D) - 1$

$\ell(D)$: dimension of complete linear system $H^0(C, \mathcal{O}_C(D))$

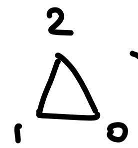
Issue: No Line Bundle - Divisor correspondence on graph

Def: We say $rk(D) = -1$ if D is not linearly equivalent to any effective divisor

Otherwise $rk(D)$ = the maximum # r such that for any effective divisor E of degree r. $D-E$ is linearly equivalent to an effective divisor

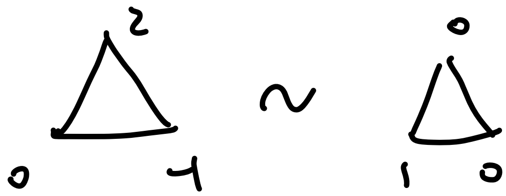
(i.e starting from chip config D on G, if our opponent is allowed to "steal" r chips, no matter what they steal we can perform chip firing moves to eliminate all debt)

Note: If one replaces G with alg. curve C, the same definition gives $r(D) = \underbrace{\ell(D)}_{\dim H^0(C, \mathcal{O}_C(D))} - 1$

Ex: ① $\mathrm{rk}\,($  $) = 2$

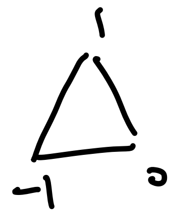
Take away 2 chips

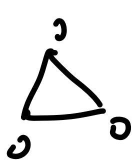
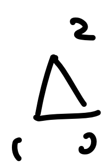
i) 

ii) 

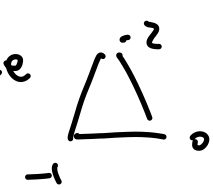
iii) 

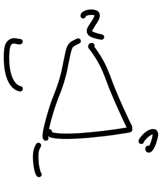
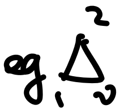
Take away 3 chips.

But  is never effective.

② $\mathrm{rk}\,($  $-$  $) = -1$

K

since  degree < 0

RR: rk() − rk() = deg  − g + 1

2 − (−1) = 3 − 1 + 1

$\boxed{\text{Yes}}$

Some consequences of RR. (all excercises)

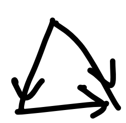
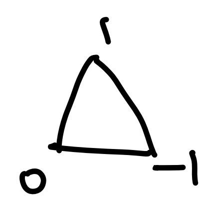
① For any D, deg D = g . rk(D) ≥ 0

i.e. Any chip config with net total g chips, can bring everyone out of debt w/ chip-firing

② Can classify all D with deg D = g−1 and rk D = −1

They come from acyclic orientations $\mathcal{O}$ of G

Given by $D_{\mathcal{O}}(v) = \text{outdeg}_{\mathcal{O}}(v) - 1$

Ex: $\mathcal{O}$ =  ⟶ $D_{\mathcal{O}}$ = 

NOTE: This is the key step in the proof.

③ For any D, $\deg D = d > 2g-2 \Rightarrow rk(D) = d-g$

$rk(K) = g-1$

④ $rk(D) \leq \frac{1}{2}\deg D$ for any $D \geq 0$

[Tropical Clifford's ineq.]

<u>More examples</u>

I. G is a tree. $g=0$

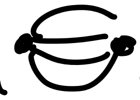
Can show that any two D, D' of same degree are linearly equivalent

$$\Rightarrow rk D = \begin{cases} \deg D & \text{if } \deg D \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

Easy to check

$$rk D - rk(\underset{\substack{\downarrow \\ \deg -2}}{K} - D) = \deg D - 0 + 1$$

II. ℓ edges



a b

$0 \leq a, b \leq \ell - 2$.

$rk(D) = \min\{a, b\}$.

$g = \ell - 1$

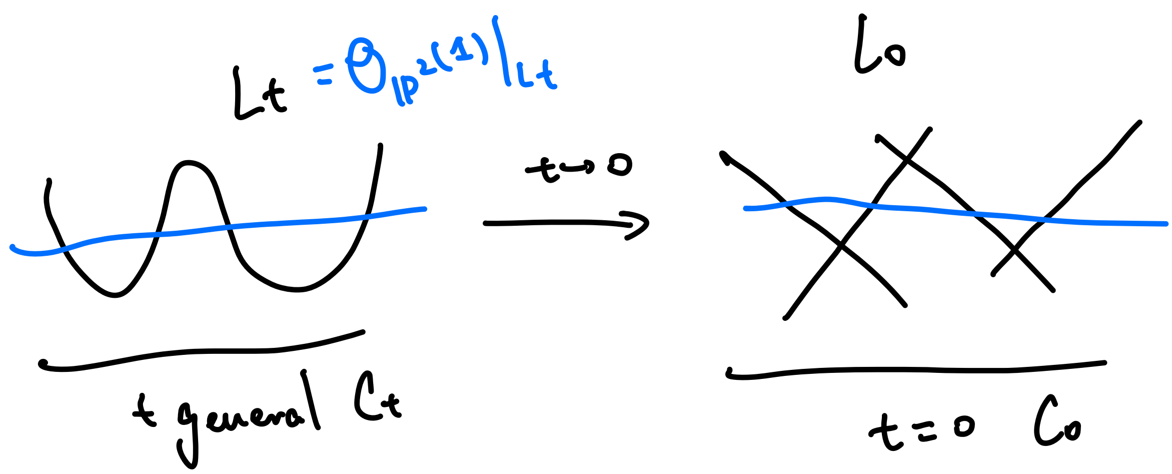
$K = (\ell-2, \ell-2)$

III. Geometric picture

Consider a family of plane curves $\mathcal{C}$ embedded in $\mathbb{P}^2$

$$\mathcal{C} = \left\{ C_t : t \cdot F(x,y,z) + xyz(x+y+z) \subseteq \mathbb{P}^2 \right\}$$

deg 4 generic homog. polynomial

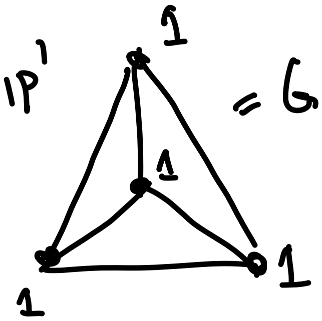


degree $= 4$

genus $= 3$ (deg-genus formula)

At the limit C_0, the information is captured by the dual graph

- each vertex is a copy of $\mathbb{P}^1$
- #edges = #intersections of two $\mathbb{P}^1$'s



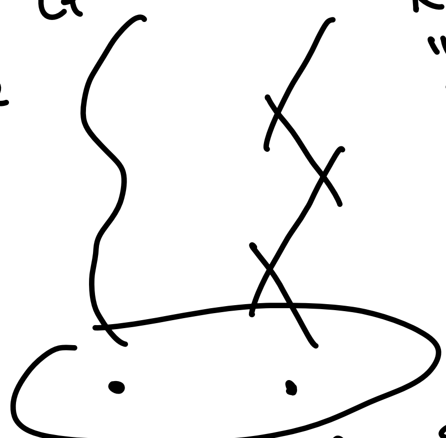
$= G$

We can also take limit of a family of line bundles L_t on C_t, the limit bundle L_0 is a divisor on dual graph G.

<u>In general</u>, $\mathcal{C}$ is a curve over Spec R
$\downarrow$
DVR

C_t C_0 k-curve
" residue field
$k = R/(m)$

K-curve
"
frac(R)



(0)   (m)   Spec R
$\downarrow$ $\downarrow$
generic pt   special point

In our example,
$R = \mathbb{C}[[t]]$
$K = \mathbb{C}((t))$
$k = \mathbb{C}$

$\mathcal{C}$ is called a "strongly semi-stable model" of C
(regular + proper, flat over R.)

Using this model, we can always extend (regular) a line bundle L_t over C_t to a line bundle $\mathcal{L}$ over $\mathcal{C}$ (by adding L_0 over C_0)

<u>Not unique!</u> If $\mathcal{L}$ works, so does

$$\mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(Y)$$

irred. comp. of C_0

$\otimes\, \mathcal{O}_{\mathcal{C}}(Y)$ on $L_0 \Longleftrightarrow$ chip-firing at vertex Y.

This extension induces a well-defined map.

$$\mathrm{Trop} : \mathrm{Pic}_K(C_t) \longrightarrow \mathrm{Pic}(G)$$

$$L_t \longmapsto L_0$$

<u>Thm</u>: [Baker's Specialization Lm] ('08)

$$\mathrm{rk}(D) \le \mathrm{rk}(\mathrm{Trop}\, D)$$

IV An application : Brill-Noether Thm

Q: If I have a curve C of genus g,

does there exist deg d map $\varphi : C \to \mathbb{P}^r$?

OR does there exist $D \in \mathrm{Div}(C)$ s.t.

$$\deg D = d, \quad \mathrm{rk}\, D = r ?$$

More algebraically, $W_d^r(C) = \{[D] \in \mathrm{Pic}^d(C) : \mathrm{rk}\, D \geq r\}$

"closed subvar." $\subseteq \mathrm{Pic}^d(C)$

Q: $W_d^r(C) \overset{?}{=} \phi$, $\dim(W_d^r(C)) = ?$

Thm: [Brill-Noether Thm]

Brill-Noether number

$$\rho = g - (r+1)(g-d+r)$$

① For any curve C

i) $\rho \geq 0 \implies W_d^r(C) \neq \phi$

ii) if $\rho \geq 0$ $\dim(W_d^r(C)) \geq \min(\rho, g)$

[G. Kempf '77] easy

[Harris-Griffiths '80]

② For a generic curve C

i) $\rho < 0 \Rightarrow W^r_d(C) = \phi$

ii) if $\rho \geq 0$, $\dim(W^r_d(C)) = \min(\rho, g)$

hard

Pf of hard part using Graphical RR

[CDPR'12]

Consider a semistable model $\mathcal{C}$

$$C_t \longrightarrow C_0 \ (\cong G)$$

$$\text{Trop}: \ \text{Pic}_K(C_t) \longrightarrow \text{Pic}(G)$$

restrict to $\quad W^r_d(C_t) \longrightarrow W^r_d(G)$

' because

$\text{rk}(\text{Trop}\, D) \geq \text{rk}\, D$.

subset of $\text{Pic}(G)$

$\{D \in \text{Pic}^d G : \text{rk}\, D \geq r\}$.

i) When $\rho<0$, if we want to show $W^r_d(G)=\phi$

only need $W^r_d(G)=\phi$

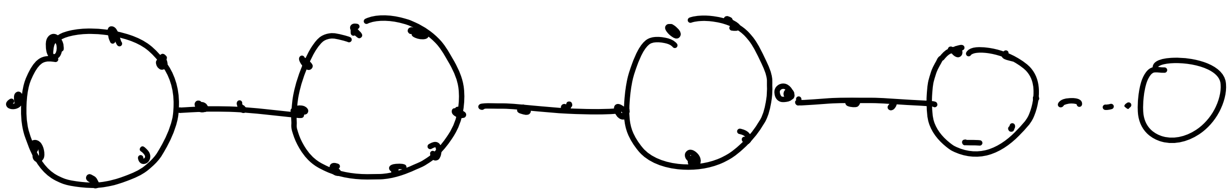
It reduces to finding a graph G

with no divisors D satisfying

$\deg D = d$ and $\operatorname{rk} D \geq r$

<u>Thm</u>: [CDPR'12] If $\rho<0$.

$G=$ chain of loops



with generic lengths $\qquad$ g loops

Then $W^r_d(G)=\phi$

<u>Note</u>: For any G and complete DVR R, there exists a strongly semi-stable model $\mathcal{C}$ over R with dual graph G.

Reason: $\overline{M}_{g,n}$ exist as a smooth DM-stack

(ii) What about $\dim(W^r_d(C)) \leq \min(\rho, g)$?

Need: dim of $W^r_d(G)$ → set

Def: $\dim W^r_d(G)$ = largest integer n s.t. for every effective E of degree $n+r$, there exist a $D \in W^r_d(G)$ s.t. $D-E$ is effective

Lm: $\dim W^r_d(G) \geq \dim W^r_d(C_t)$

→ similarly reduces to a question on graphs

→ G = chain of loops still work.

Alg $\Rightarrow$ Combinatorics too!

Easy part $W^r_d(C) \neq \emptyset$ for any curve C when $\rho \geq 0$

Cor: For any G of genus g, any r, d s.t. $\rho \geq 0$.

Then there exists a divisor D on * G s.t. $\frac{1}{e}G$.

$\deg D = d$, $\operatorname{rk} D \geq r$

$\boxed{\text{No known combinatorial proof}}$

V Real version: Metric Graph

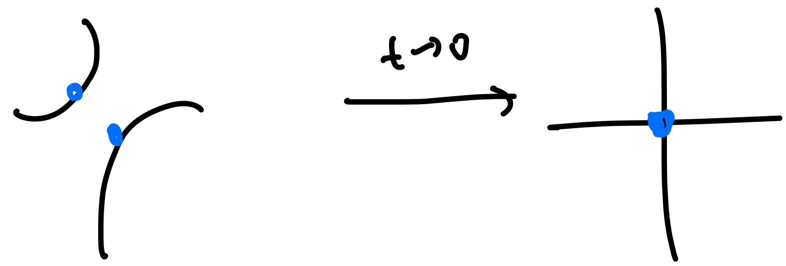
Issue: K is not algebraically closed

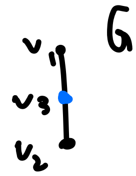
$$\mathrm{Trop}: \mathrm{Pic}_K(C_t) \longrightarrow \mathrm{Pic}(G)$$

Can only show this is empty

but there might be divisors over finite extension K'/K

Ex: $C_t = (xy = tz^2)$ $\qquad$ $C_0 = (xy = 0)$





divisor along $x = y$.

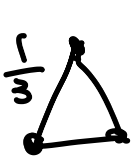
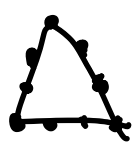
$D = P + Q$ $\qquad$ $K' = \mathbb{C}((\sqrt{t}))$ $\qquad$ $\mathrm{trop}\, D = 2v_3$.

$P \downarrow (\sqrt{t} : \sqrt{t} : 1)$, $\quad Q \downarrow (-\sqrt{t}, -\sqrt{t}, 1)$

Thm: If K' is a finite extension over K and let $C_{K'} = C \times_K K'$. If $\mathcal{C}$ is a semistable model of C with dual graph G then there is a model $\mathcal{C}'$ of $C_{K'}$ over R' with dual graph $\frac{1}{e} G$

$e = [K' : K]$

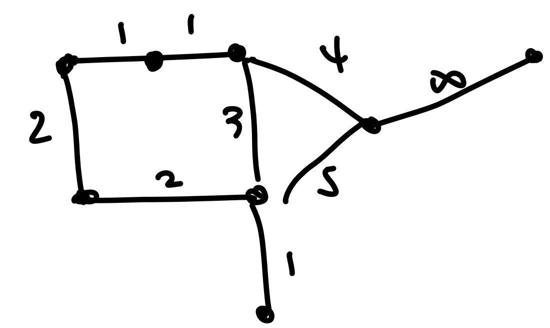
e-subdivision

e.g.: $\frac{1}{3}$  $=$ 

If one consider all extensions, i.e. $\overline{K}$, motivates:

<u>Def</u>: A metric graph is a finite graph such that each edge is homeomorphic to an interval $[0,a]$ where $a \in [0,\infty]$ and $a=\infty$ only occurs when one endpoint is a leaf

<u>Ex</u>:

$G =$ 

Divisor $D = \sum_{v \in G} D(v) \cdot v$ where $D(v) \in \mathbb{Z}$

and $D(v) \neq 0$ for finitely many points on G. (can choose points on edge)

<u>Note</u>: • Can similarly define everything.

RR still holds.

- Compatible with original definition if no self loops + edge length 1.

- Extends

$$\mathrm{Trop}: \mathrm{Pic}_K(C_t) \to \mathrm{Pic}(G)$$

to $\mathrm{Trop}: \mathrm{Pic}_{\overline{K}}(C_t) \to \mathrm{Pic}(G)$

metric graph version.