Balanced Shifted Tableaux

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[1] We are not related

Standard Young tableaux

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0)$ be a partition.

A **standard Young tableau** of shape λ is a filling of λ using $1, \ldots, |\lambda|$ such that each row and each column form increasing sequences.

Example: standard young tableaux of shape $\lambda = (4, 3, 2)$

1	2	5	9
3	6	7	
4	8		

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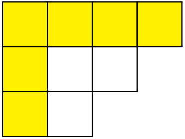
For each box (i, j) in its Young diagram, let its **hook** $H_\lambda(i, j)$ consist of all the boxes directly to the right or the bottom of (i, j), including itself.



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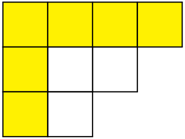

Theorem (Hook length formula)

The number of standard Young tableaux of shape λ equals

$$f^\lambda = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} |H_\lambda(i,j)|}.$$

Balanced tableaux

For a box $(i, j) \in \lambda$, let $\mathrm{rk}_\lambda(i, j)$ be the size of the right arm of $H_\lambda(i, j)$, i.e. the number of boxes to the right of (i, j), including itself.

	1	2	3	4
1	3	7	4	2
2	5	8	6	
3	1	9		

$\mathrm{rk}_\lambda(1, 1) = 4$

A **balanced tableau** of shape λ is a filling T of λ using $1, \ldots, |\lambda|$ such that $T(i, j)$ is the $\mathrm{rk}_\lambda(i, j)$-th largest entry in its hook.

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Theorem (Edelman-Greene 1987)

For a partition λ, the number of balanced tableau of shape λ equals the number of standard Young tableaux of shape λ.

Shifted shapes

Let $\lambda = (\lambda_1 > \cdots > \lambda_d)$ be a **strict** partition, which corresponds to a **shifted shape** by shifting the i-th row i steps to the right.

A **standard Young tableau** of shifted shape λ is a filling of λ using $1, \ldots, |\lambda|$ that is increasing in each row and column.

Example: SYT of shifted shape $\lambda = (6, 2, 1)$

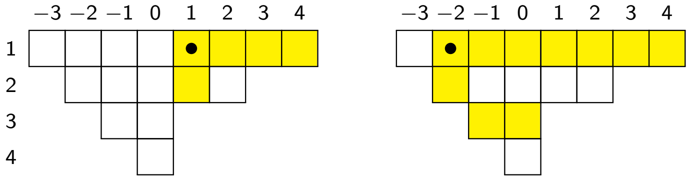
1	2	3	5	6	9
	4	7			
		8			

Let $\mathrm{SYT}(\lambda)$ be the set of standard Young tableaux of λ.

Hook length formula for shifted shapes

Let $\lambda = (\lambda_1 > \cdots > \lambda_d)$ be a shifted shape. The hook $H_\lambda(i, j)$ contains:

- boxes to the right and below, if $j \geq 0$;
- boxes to the right and below, and then turn again to the right with a "broken leg", if $j < 0$.



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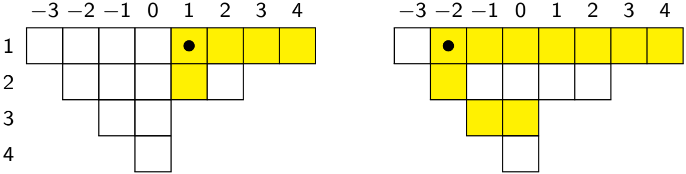


Theorem (Hook length formula for shifted shapes)

The number of standard Young tableaux of shifted shape λ equals

$$|\operatorname{SYT}(\lambda)| = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} |H_\lambda(i,j)|}.$$

Balanced Shifted Tableaux?

- If we define $\text{rk}_\lambda(i,j) =$ size of right arm of $H_\lambda(i,j)$, and define balanced shifted tableaux analogously, then they are **not** equinumerous to standard shifted tableaux.

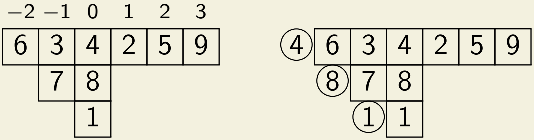
Balanced Shifted Tableaux?

- If we define $\text{rk}_\lambda(i,j) =$ size of right arm of $H_\lambda(i,j)$, and define balanced shifted tableaux analogously, then they are **not** equinumerous to standard shifted tableaux.
- We make a definition for balanced shifted tableaux as close to this idea as possible, and show that they are equinumerous to standard shifted tableaux.
- Specifically, we incorporate two fixes:
  - "Extended" hooks $\tilde{H}_\lambda(i,j)$
  - A new rank function $\text{rk}_\lambda(i,j)$

Extended hooks

For a filling B of λ, we copy the 0^{th} column of B and paste it to the left of λ. We call this the **extended filling** $\tilde{B}$.

Example: extended filling



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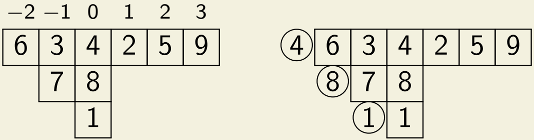
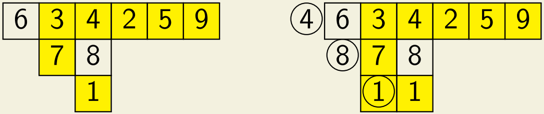


The **extended hook** $\tilde{H}_\lambda(i,j)$ is the same as the original hook $H_\lambda(i,j)$ if $j \geq 0$, and is $H_\lambda(i,j) \cup \{\text{one extra corner box}\}$ if $j < 0$.

Example: extended hook

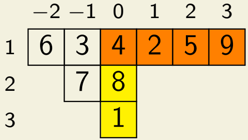


A new rank function

Define the following rank function

- $\mathrm{rk}_\lambda(i,j) = \#$boxes in row i of $H_\lambda(i,j)$, $j \geq 0$,
- $\mathrm{rk}_\lambda(i,j) = \#$boxes with non-negative column index of $H_\lambda(i,j)$, $j < 0$.

Example: The new rank function



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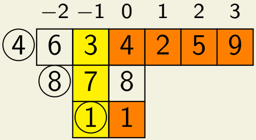

$rk_\lambda(1,-1) = 5$

Balanced Shifted Tableaux

Let $\mathrm{rk}_\lambda(i,j)$ be defined as above.

Definition (G-Gao-Gao 2022)

A **balanced shifted tableau** of shape λ is a filling B of λ using $1,\ldots,|\lambda|$ such that for all $(i,j) \in \lambda$, $B(i,j)$ is the $\mathrm{rk}_\lambda(i,j)$-th largest entry in the extended hook $\tilde{H}_\lambda(i,j)$.

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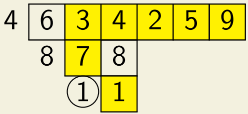
Let $\mathrm{BS}(\lambda)$ be the set of balanced shifted tableaux of shape λ.

Example: a balanced shifted tableau



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Example: a balanced shifted tableau

4	6	3	4	2	5	9
	8	7	8			
		(1)	1			

Theorem (G-Gao-Gao 2022)

For a shifted shape λ, $|\text{SYT}(\lambda)| = |\text{BS}(\lambda)|$.

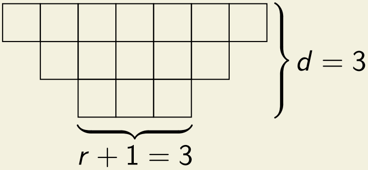
Proof sketch

Our proof is bijective, with the following strategy:

- We first provide a bijection for the trapezoid shape $Z(d, r)$;
- For general shape λ, pad it to $Z(d, r)$ and apply the bijection for $Z(d, r)$, and finally restrict to shape λ.

The trapezoid $Z(d, r)$

$Z(3, 2) =$



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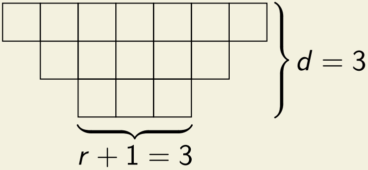

$$\mathrm{BS}(\lambda) \xrightarrow{\text{pad}} \mathrm{BS}(Z(d,r))|_\lambda \xrightarrow{\sim} \mathrm{Red}(w^{(d,r)})|_\lambda \xrightarrow{\sim} \mathrm{SYT}(Z(d,r))|_\lambda$$

$$\mathrm{SYT}(\lambda) \xleftarrow{\text{restrict}} \mathrm{SYT}(Z(d,r))|_\lambda$$

An example

Let's start with a balanced shifted tableau

$B =$

6	3	4	1	5	9
	7	8			
		2			

We have $\lambda = (6, 2, 1)$ and choose trapezoid $Z(d, r) = Z(3, 2)$.

$$\mathrm{BS}(\lambda) \xrightarrow{\text{pad}} \mathrm{BS}(Z(d,r))|_\lambda \xrightarrow{\sim} \mathrm{Red}(w^{(d,r)})|_\lambda$$

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Step 1: $\mathrm{BS}(\lambda) \to \mathrm{BS}(Z(d,r))|_\lambda$

We pad $B \in \mathrm{BS}(\lambda)$ to shape $Z(d,r)$ by adding boxes with entries $|\lambda|+1, |\lambda|+2, \ldots$ from left to right, top to bottom. Each time we add a box to column j, interchange columns j and $j+1$.

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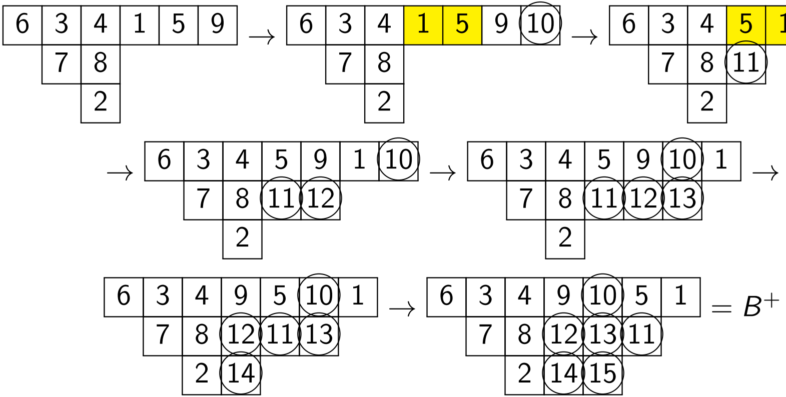

Step 2: $\mathrm{BS}(Z(d,r)) \to \mathrm{Red}(w^{(d,r)})$ via reflection order

A **signed permutation** $\sigma \in W(B_n)$ is a permutation on $1, \ldots, n, \bar{1}, \ldots, \bar{n}$ such that $\sigma(i) = -\sigma(\bar{i})$ (here $\bar{i} = -i$). We denote σ by $\sigma(1)\sigma(2)\cdots\sigma(n)$. For example, $\sigma = 1\bar{3}42 \in W(B_4)$.

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Any $\sigma \in W(B_n)$ can be written as a product of **simple reflections** s_0 (which negates 1^{st} entry), and s_i (which swaps i^{th} and $(i+1)^{\text{th}}$ entry). A **reduced word** of σ is such a product of minimal length. For example, $\sigma = s_2 s_1 s_0 s_3 s_1$:

$$1234 \xrightarrow{s_2} 1324 \xrightarrow{s_1} 3124 \xrightarrow{s_0} \bar{3}124 \xrightarrow{s_3} \bar{3}142 \xrightarrow{s_1} 1\bar{3}42 = \sigma.$$

$\mathbf{a} = (2,1,0,3,1) \in \mathrm{Red}(\sigma)$ is a reduced word of σ.

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$\mathbf{a} = (2,1,0,3,1) \in \mathrm{Red}(\sigma)$ is a reduced word of σ.

Define a signed permutation $w^{(d,r)} \in W(B_{d+r})$ by

$$w^{(d,r)} = (d+1)(d+2)\cdots(d+r)\bar{1}\bar{2}\cdots\bar{r}.$$

For example, $w^{(3,2)} = 45\bar{1}\bar{2}\bar{3}$.

Step 2: $\mathrm{BS}(Z(d,r)) \to \mathrm{Red}(w^{(d,r)})$ via reflection order

We provide a labeling of $Z(3,2)$ by positive roots in $\mathrm{Inv}(w^{(3,2)})$:

$2e_3$	e_3+e_2	e_3+e_1	e_3	e_4+e_3	e_5+e_3	e_3-e_1	e_3-e_2
	$2e_2$	e_2+e_1	e_2	e_4+e_2	e_5+e_2	e_2-e_1	
		$2e_1$	e_1	e_4+e_1	e_5+e_1		

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	$2e_2$	e_2+e_1	e_2	e_4+e_2	e_5+e_2	e_2-e_1	
		$2e_1$	e_1	e_4+e_1	e_5+e_1		

Start with a reduced word $\mathbf{a} = (2,0,1,0,\ldots)$ of $w^{(3,2)} = 45\bar{1}\bar{2}\bar{3}$

$$12345 \xrightarrow{e_3-e_2} 13245 \xrightarrow{e_1} \bar{1}3245 \xrightarrow{e_3+e_1} 3\bar{1}245 \xrightarrow{e_3} \bar{3}\bar{1}245 \longrightarrow \cdots$$

	3	4				1
		2				

Proposition (G-Gao-Gao 2022)

The reflection order bijects $\mathrm{Red}(w^{(d,r)})$ *to* $\mathrm{BS}(Z(d,r))$.

Step 2: $\mathrm{BS}(Z(d,r)) \to \mathrm{Red}(w^{(d,r)})$ via reflection order

$$
B^+ = \begin{array}{|c|c|c|c|c|c|c|}
\hline
6 & 3 & 4 & 9 & \textcircled{10} & 5 & 1 \\
\hline
\multicolumn{1}{c|}{} & 7 & 8 & \textcircled{12} & \textcircled{13} & \textcircled{11} & \multicolumn{1}{|c}{} \\
\cline{2-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & \textcircled{14} & \textcircled{15} & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} \\
\cline{3-5}
\end{array}
$$

gives a reflection order

$$
12345 \xrightarrow{e_3-e_2} 13245 \xrightarrow{e_1} \bar{1}3245 \xrightarrow{e_3+e_1} 3\bar{1}245 \xrightarrow{e_3} \bar{3}\bar{1}245 \xrightarrow{e_3-e_1} \bar{1}\bar{3}245
$$

$$
\xrightarrow{e_3+e_2} \bar{1}2\bar{3}45 \xrightarrow{e_2+e_1} 2\bar{1}\bar{3}45 \xrightarrow{e_2} \bar{2}\bar{1}\bar{3}45 \xrightarrow{e_4+e_3} \bar{2}\bar{1}4\bar{3}5 \xrightarrow{e_5+e_3} \bar{2}\bar{1}45\bar{3}
$$

$$
\xrightarrow{e_2-e_1} \bar{1}\bar{2}45\bar{3} \xrightarrow{e_4+e_2} \bar{1}4\bar{2}5\bar{3} \xrightarrow{e_5+e_2} \bar{1}45\bar{2}\bar{3} \xrightarrow{e_4+e_1} 4\bar{1}5\bar{2}\bar{3} \xrightarrow{e_5+e_1} 45\bar{1}\bar{2}\bar{3},
$$

for which we read off

$$
\mathbf{a} = 201012103412312 \in \mathrm{Red}(w^{(3,2)}).
$$

Step 3: $\text{Red}(w^{(d,r)}) \to \text{SYT}(Z(d,r))$ via Kraśkiewicz's insertion

Kraśkiewicz's insertion is an algorithm that bijects reduced word $\mathbf{a} \in \text{Red}(\sigma)$ to a pair of shifted tableaux $(P(\mathbf{a}), Q(\mathbf{a}))$ of the same shape, where

- $P(\mathbf{a})$ is a *standard decomposition tableaux* of σ;
- $Q(\mathbf{a})$ is a standard shifted tableaux called the *recording tableaux*.

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For example, Kraśkiewicz's insertion of $\mathbf{a} = 201012103412312$ gives

$P^{(0)} = \emptyset$ $\qquad$ $Q^{(0)} = \emptyset$

$P^{(3)} =$

2	0	1

$Q^{(3)} =$

1	2	3

$P^{(4)} =$

2	1	0
	0	

$Q^{(4)} =$

1	2	3
	4	

$\vdots$ $\qquad$ $\vdots$

$P^{(15)} =$

4	3	0	1	2	3	4
	3	0	1	2	3	
		0	1	2		

$Q^{(15)} =$

1	2	3	5	6	9	10
	4	7	11	12	13	
		8	14	15		

By restricting to the recording tableaux $Q(\mathbf{a})$, Kraśkiewicz's insertion gives a bijection $\mathbf{a} \mapsto Q(\mathbf{a})$ between $\text{Red}(w^{(d,r)})$ and $\text{SYT}(Z(d,r))$.

Step 4: $\mathrm{SYT}(Z(d,r))|_\lambda \to \mathrm{SYT}(\lambda)$

For $T^+ \in \mathrm{SYT}(Z(d,r))$, we simply delete extraneous boxes from T^+ to obtain $T \in \mathrm{SYT}(\lambda)$.

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For example,

$$T^+ = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 9 & \textcircled{10} \\ \hline \multicolumn{1}{c|}{} & 4 & 7 & \textcircled{11} & \textcircled{12} & \textcircled{13} & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & \textcircled{14} & \textcircled{15} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-5} \end{array}, \text{ then } T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 9 \\ \hline \multicolumn{1}{c|}{} & 4 & 7 & \multicolumn{3}{c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & \multicolumn{3}{c}{} \\ \cline{3-3} \end{array}.$$

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For example,

$T^+ =$

1	2	3	5	6	9	⑩
	4	7	⑪	⑫	⑬	
		8	⑭	⑮		

, then $T =$

1	2	3	5	6	9
	4	7			
		8			

.

We have completed the bijection

Example: the bijection $\mathrm{BS}(\lambda) \to \mathrm{SYT}(\lambda)$

$B =$

6	3	4	1	5	9
	7	8			
		2			

$T =$

1	2	3	5	6	9
	4	7			
		8			

Thanks!

Thank you for listening!

Some remarks

There are other notions of "balanced tableaux" in the literature, including

- "balanced labeling" by Fomin-Greene-Reiner-Shimozono,
- and its type B analogue by Zachary Hamaker.

However,

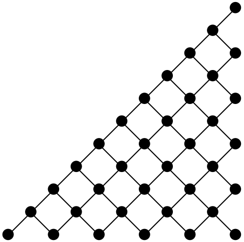
- The above definitions are defined on Rothe diagrams.
- Our definition focuses on tableau and the hook length formula.

Some remarks

What about other root systems?

- Balanced tableaux $\Longleftrightarrow$ reflection order;
- Standard tableaux $\Longleftrightarrow$ linear extension of root poset.

Example: type B_n root poset ($\cong Z(n, 0)$)



Conjecture: # reflection orders = # linear extension of root poset?

Some remarks

Unfortunately, $\# \operatorname{Red}(w_0(D_4)) = 2316$ and the number of linear extensions of the root poset is $e(\Phi(D_4)^+) = 2400$.

These two quantities also fail to be equal in F_4.

Conjecture (Stanley 1984)

For any Coxeter group W and $J \subset S$,

$$\# \operatorname{Red}(w_0^J) \leq e(\Phi_J^+).$$