Balanced Shifted Tableaux

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[1] We are not related

Standard Young tableaux

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0)$ be a partition.

A **standard Young tableau** of shape λ is a filling of λ using $1, \ldots, |\lambda|$ such that each row and each column form increasing sequences.

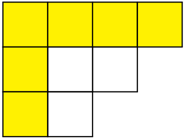
Example: standard young tableaux of shape $\lambda = (4, 3, 2)$

1	2	5	9
3	6	7	
4	8		

Standard Young tableaux

Question: How many standard Young tableaux of shape λ are there?

For each box (i,j) in its Young diagram, let its **hook** $H_\lambda(i,j)$ consist of all the boxes directly to the right or the bottom of (i,j), including itself.



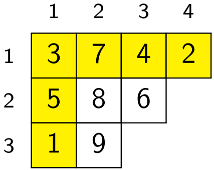
Theorem (Hook length formula)

The number of standard Young tableaux of shape λ equals

$$f^\lambda = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} |H_\lambda(i,j)|}.$$

Balanced tableaux

For a box $(i, j) \in \lambda$, let $\mathrm{rk}_\lambda(i, j)$ be the size of the right arm of $H_\lambda(i, j)$, i.e. the number of boxes to the right of (i, j), including itself.



	1	2	3	4
1	3	7	4	2
2	5	8	6	
3	1	9		

$\mathrm{rk}_\lambda(1, 1) = 4$

A **balanced tableau** of shape λ is a filling T of λ using $1, \ldots, |\lambda|$ such that $T(i, j)$ is the $\mathrm{rk}_\lambda(i, j)$-th largest entry in its hook.

Theorem (Edelman-Greene 1987)

For a partition λ, the number of balanced tableau of shape λ equals the number of standard Young tableaux of shape λ.

A counting problem

Fix λ. Let $\mathrm{rk}_\lambda : \lambda \to \mathbb{Z}_{>0}$ be a function on the boxes in λ.

Define a **balanced tableau with respect to** rk_λ to be a filling T of λ using $1, \ldots, |\lambda|$ such that $T(i,j)$ is the $\mathrm{rk}_\lambda(i,j)$-th largest entry in its hook.

- $\mathrm{rk}_\lambda =$ size of the right arm gives balanced tableaux
- $\mathrm{rk}_\lambda = \mathbf{1}$ gives standard Young tableaux

Question

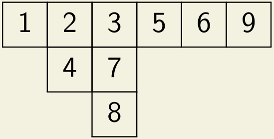
Given λ, what are all the possible functions rk_λ such that the number of balanced tableaux with respect to rk_λ can be enumerated by the hook length formula?

Shifted shapes

Let $\lambda = (\lambda_1 > \cdots > \lambda_d)$ be a **strict** partition, which corresponds to a **shifted shape** by shifting the *i*-th row *i* steps to the right.

A **standard Young tableau** of shifted shape λ is a filling of λ using $1, \ldots, |\lambda|$ that is increasing in each row and column.

Example: SYT of shifted shape $\lambda = (6, 2, 1)$

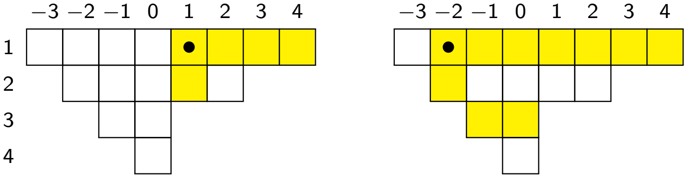


Let $\mathrm{SYT}(\lambda)$ be the set of standard Young tableaux of λ.

Hook length formula for shifted shapes

Let $\lambda = (\lambda_1 > \cdots > \lambda_d)$ be a shifted shape. The hook $H_\lambda(i,j)$ contains:

- boxes to the right and below, if $j \geq 0$;
- boxes to the right and below, and then turn again to the right with a "broken leg", if $j < 0$.



Theorem (Hook length formula for shifted shapes)

The number of standard Young tableaux of shifted shape λ equals

$$|\operatorname{SYT}(\lambda)| = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} |H_\lambda(i,j)|}.$$

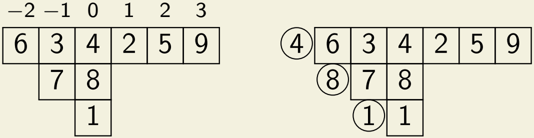
Balanced Shifted Tableaux?

- If we define $\mathrm{rk}_\lambda(i,j) =$ size of right arm of $H_\lambda(i,j)$, and define balanced shifted tableaux analogously, then they are **not** equinumerous to standard shifted tableaux.
- We make a definition for balanced shifted tableaux as close to this idea as possible, and show that they are equinumerous to standard shifted tableaux.
- Specifically, we incorporate two fixes:
  - "Extended" hooks $\tilde{H}_\lambda(i,j)$
  - A new rank function $\mathrm{rk}_\lambda(i,j)$

Extended hooks

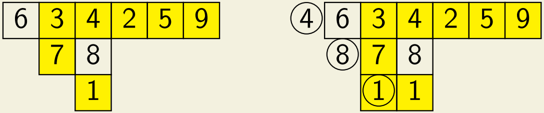
For a filling B of λ, we copy the 0^{th} column of B and paste it to the left of λ. We call this the **extended filling** $\tilde{B}$.

Example: extended filling



The **extended hook** $\tilde{H}_\lambda(i,j)$ is the same as the original hook $H_\lambda(i,j)$ if $j \geq 0$, and is $H_\lambda(i,j) \cup \{\text{one extra corner box}\}$ if $j < 0$.

Example: extended hook

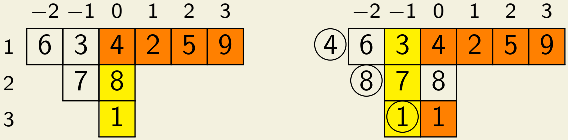


A new rank function

Define the following rank function

- $\mathrm{rk}_\lambda(i,j) = \#$boxes in row i of $H_\lambda(i,j)$, $j \geq 0$,
- $\mathrm{rk}_\lambda(i,j) = \#$boxes with non-negative column index of $H_\lambda(i,j)$, $j < 0$.

Example: The new rank function



$rk_\lambda(1,0) = 4$ $rk_\lambda(1,-1) = 5$

Balanced Shifted Tableaux

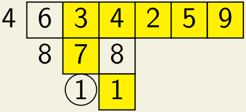
Let $\mathrm{rk}_\lambda(i,j)$ be defined as above.

Definition (G-Gao-Gao 2022)

A **balanced shifted tableau** of shape λ is a filling B of λ using $1,\ldots,|\lambda|$ such that for all $(i,j)\in\lambda$, $B(i,j)$ is the $\mathrm{rk}_\lambda(i,j)$-th largest entry in the extended hook $\tilde{H}_\lambda(i,j)$.

Let $\mathrm{BS}(\lambda)$ be the set of balanced shifted tableaux of shape λ.

Example: a balanced shifted tableau



4 6 3 4 2 5 9
8 7 8
① 1

Theorem (G-Gao-Gao 2022)

For a shifted shape λ, $|\mathrm{SYT}(\lambda)| = |\mathrm{BS}(\lambda)|$.

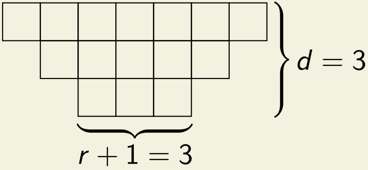
Proof sketch

Our proof is bijective, with the following strategy:

- We provide a bijection for the trapezoid $Z(d,r)$, by relating them with the set of *reduced words* of a certain *signed permutation* $w^{(d,r)}$.
- For any λ, pad it to $Z(d,r)$ and apply the bijection for $Z(d,r)$.

The trapezoid $Z(d,r)$

$Z(3,2) =$



$$\begin{tikzcd}
\mathrm{SYT}(\lambda) \arrow[r, leftrightarrow] & \mathrm{SYT}(Z(d,r))|_\lambda \arrow[r, leftrightarrow] \arrow[d, hook] & \mathrm{Red}(w^\lambda) \arrow[r, leftrightarrow] \arrow[d, hook] & \mathrm{BS}(Z(d,r))|_\lambda \arrow[r, leftrightarrow] \arrow[d, hook] & \mathrm{BS}(\lambda) \\
 & \mathrm{SYT}(Z(d,r)) \arrow[r, leftrightarrow] & \mathrm{Red}(w^{(d,r)}) \arrow[r, leftrightarrow] & \mathrm{BS}(Z(d,r)) &
\end{tikzcd}$$

Type B_n root system

Let $W(B_n)$ be the set of permutations w on $1, \dots, n, \bar{1}, \dots, \bar{n}$ such that $w(i) = -w(\bar{i})$ (here $\bar{i} = -i$). We call w a **signed permutation**, denoted by its one-line notation $w(1)w(2)\cdots w(n)$.

$W(B_n)$ is generated by **simple reflections** s_0 (which negates $w(1)$), and s_i (which swaps $w(i)$ and $w(i+1)$) for $1 \le i \le n-1$.

For $w \in W(B_n)$, its **length** $\ell(w)$ is the minimal ℓ such that $w = s_{a_1} \cdots s_{a_\ell}$ is a product of ℓ simple reflections. Such a word $\mathbf{a} = (a_1, \dots, a_\ell)$ is a **reduced word** of w.

Example: reduced word of w

$w = 1\bar{3}42 \in W(B_4)$:

$$1234 \xrightarrow{s_2} 1324 \xrightarrow{s_1} 3124 \xrightarrow{s_0} \bar{3}124 \xrightarrow{s_3} \bar{3}142 \xrightarrow{s_1} 1\bar{3}42 = w.$$

Reduced word: $(2, 1, 0, 3, 1) \in \mathrm{Red}(w)$.

Type B_n root system

The **type B_n root system** is a set of vectors in $\mathbb{R}^n$ given by $\Phi(B_n) = \{\pm e_j \pm e_i \mid 1 \le i < j \le n\} \cup \{\pm e_i \mid 1 \le i \le n\}$ (here e_i is the standard basis).

We can partition $\Phi(B_n)$ into **positive roots** $\Phi^+(B_n)$ and negative roots $\Phi^-(B_n)$. Here $\Phi^+(B_n) = \{e_j \pm e_i \mid 1 \le i < j \le n\} \cup \{e_i \mid 1 \le i \le n\}$.

$W(B_n)$ naturally acts on $\Phi(B_n)$ by permuting/negating the entries. For $w \in W(B_n)$, its **inversion set** $\mathrm{Inv}(w)$ is the set of positive roots $\alpha \in \Phi^+(B_n)$ such that $w^{-1}\alpha$ is a negative root.

Fact: $|\mathrm{Inv}(w)| = \ell(w)$.

Reflection order of positive roots

Example: inversion set and reflection order

$w = 1\bar{3}42 \in W(B_4)$, $\mathbf{a} = (2,1,0,3,1) \in \mathrm{Red}(w)$:

$$1234 \xrightarrow{s_2} 1324 \xrightarrow{s_1} 3124 \xrightarrow{s_0} \bar{3}124 \xrightarrow{s_3} \bar{3}142 \xrightarrow{s_1} 1\bar{3}42 = w.$$

If we read the values instead:

$$1234 \xrightarrow{e_3-e_2} 1324 \xrightarrow{e_3-e_1} 3124 \xrightarrow{e_3} \bar{3}124 \xrightarrow{e_4-e_2} \bar{3}142 \xrightarrow{e_3+e_1} 1\bar{3}42 = w.$$

$\mathrm{Inv}(w) = \{e_3 - e_2, e_3 - e_1, e_3, e_4 - e_2, e_3 + e_1\}$

Notice that each reduced word corresponds to an ordering of $\mathrm{Inv}(w)$. We call this the **reflection order** of the reduced word.

Proposition (Björner 1984)

Let γ be an ordering of $\mathrm{Inv}(w)$. Then γ is a reflection order if and only if for all the triples $\alpha, \beta, \alpha+\beta \in \Phi^+$ such that $\alpha, \alpha+\beta \in \mathrm{Inv}(w)$,

1. *if $\beta \in \mathrm{Inv}(w)$, then $\alpha+\beta$ appears in between α and β.*
2. *if $\beta \notin \mathrm{Inv}(w)$, then α appears before $\alpha+\beta$ in this sequence.*

$\mathrm{Red}(w^{(d,r)}) \to \mathrm{BS}(Z(d,r))$ via reflection order

Define a signed permutation $w^{(d,r)} \in W(B_{d+r})$ by

$$w^{(d,r)} = (d+1)(d+2)\cdots(d+r)\bar{1}\bar{2}\cdots\bar{r}.$$

For example, $w^{(3,2)} = 45\bar{1}\bar{2}\bar{3}$.

Proposition (G-Gao-Gao 2022)

The reflection order bijects $\mathrm{Red}(w^{(d,r)})$ *to* $\mathrm{BS}(Z(d,r))$.

$\mathrm{Red}(w^{(d,r)}) \to \mathrm{BS}(Z(d,r))$ via reflection order

We provide an extended label of $Z(d,r)$ by roots in $\mathrm{Inv}(w^{(d,r)})$:

Example: label of $Z(3,2)$ by roots

$2e_3$	e_3+e_2	e_3+e_1	e_3	e_4+e_3	e_5+e_3	e_3-e_1	e_3-e_2
	$2e_2$	e_2+e_1	e_2	e_4+e_2	e_5+e_2	e_2-e_1	
		$2e_1$	e_1	e_4+e_1	e_5+e_1		

Consider a reduced word $\mathbf{a} = (2, 0, 1, 0, \dots$ with reflection order

$$12345 \xrightarrow{e_3-e_2} 13245 \xrightarrow{e_1} \bar{1}3245 \xrightarrow{e_3+e_1} 3\bar{1}245 \xrightarrow{e_3} \bar{3}\bar{1}245 \longrightarrow \cdots$$

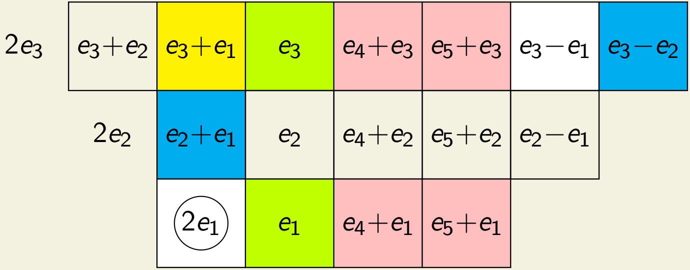
	3	4				1
		2				

$\mathrm{Red}(w^{(d,r)}) \to \mathrm{BS}(Z(d,r))$ via reflection order

Why does a reflection order give a balanced shifted tableaux?

Recall that in a reflection order, $\alpha + \beta$ appears between α and β.

Example: label of $Z(3,2)$ by roots



$2e_3$	e_3+e_2	e_3+e_1	e_3	e_4+e_3	e_5+e_3	e_3-e_1	e_3-e_2
	$2e_2$	e_2+e_1	e_2	e_4+e_2	e_5+e_2	e_2-e_1	
		$\textcircled{$2e_1$}$	e_1	e_4+e_1	e_5+e_1		

The other direction $\mathrm{BS}(Z(d,r)) \to \mathrm{Red}(w^{(d,r)})$ is non-trivial.

$\mathrm{Red}(w^{(d,r)}) \to \mathrm{SYT}(Z(d,r))$ via Kraśkiewicz's insertion

Let $w \in W(B_n)$. Kraśkiewicz's insertion is an algorithm that maps a reduced word $\mathbf{a} = (a_1, \ldots, a_\ell) \in \mathrm{Red}(w)$ to a pair of shifted tableaux $(P(\mathbf{a}), Q(\mathbf{a}))$ of the same shape, where

- $Q(\mathbf{a})$ is a standard tableaux;
- $P(\mathbf{a})$ is a **standard decomposition tableaux (SDT)** of w such that
    - each row of $P(\mathbf{a})$ is a unimodal sequence $r_1 > r_2 > \cdots > r_k < \cdots < r_m$;
    - reading the entries of $P(\mathbf{a})$ from left-to-right, bottom-to-top gives us a reduced word of w.

We start with $(P^{(0)}, Q^{(0)}) = (\emptyset, \emptyset)$, and insert one element at a time.

$$(P^{(i)}, Q^{(i)}) := (P^{(i-1)}, Q^{(i-1)}) \leftarrow a_i.$$

$\mathrm{Red}(w^{(d,r)}) \to \mathrm{SYT}(Z(d,r))$ via Kraśkiewicz's insertion

First row of $P^{(i-1)}$:

$$R = \underbrace{r_1 > r_2 > \cdots > r_k}_{R^{\downarrow}} < \underbrace{r_{k+1} < \cdots < r_m}_{R^{\uparrow}}$$

Kraśkiewicz's insertion $P^{(i-1)} \leftarrow a_i$:

- Insert a_i into the increasing part of the first row $R^{\uparrow}$, and pop the smallest number $b \geq a_i$. If $b = a_i$ then $b \to b+1$;
- Insert b into the decreasing part of the first row $R^{\downarrow}$, and pop the largest number $c \geq b$. If $c = b$ then $c \to c-1$;
- Insert c into the next row, repeat;
- (Special) If R contains 101 as a subsequence and $a_i = 0$, do nothing and insert 0 to the next row.

Record the change of shape in $Q^{(i-1)} \to Q^{(i)}$.

An example: Kraśkiewicz's insertion

Let $\mathbf{a} = (3, 1, 2, 1, 0, 3, 4, 3)$.

$$P^{(3)} = \begin{array}{|c|c|c|} \hline 3 & 1 & 2 \\ \hline \end{array} \qquad Q^{(3)} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}$$

$$P^{(4)} = \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \qquad Q^{(4)} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}$$

$$P^{(7)} = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 2 & 1 & 0 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array} \qquad Q^{(7)} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 7 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}$$

$$P^{(8)} = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 2 & 1 & 0 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 1 & 3 & \multicolumn{3}{c}{} \\ \cline{2-3} \end{array} \qquad Q^{(8)} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 7 \\ \hline \multicolumn{1}{c|}{} & 4 & 8 & \multicolumn{3}{c}{} \\ \cline{2-3} \end{array}$$

Kraśkiewicz's insertion

Theorem (Kraśkiewicz 1989)

The Kraśkiewicz's insertion gives a bijection between $\{\mathbf{a} \in \mathrm{Red}(w)\}$ *and the pairs of tableaux* $(P(\mathbf{a}), Q(\mathbf{a}))$ *where* $P(\mathbf{a})$ *is a standard decomposition tableaux of* w *and* $Q(\mathbf{a})$ *is a standard tableaux of the same shape.*

A signed permutation $w \in W(B_n)$ is **vexillary** if $\mathrm{SDT}(w)$ consists of exactly one shifted tableau. We denote this tableau as $P(w)$.

There is a nice criteria for when a signed permutation is vexillary.

Theorem (Billey-Lam 1998)

A signed permutation $w \in W(B_n)$ *is vexillary if and only if* w *pattern avoids* 2143 *as a permutation in* $\mathfrak{S}_{2n}$*.*

Corollary

$w^{(d,r)}$ *is vexillary.*

$\mathrm{Red}(w^{(d,r)}) \to \mathrm{SYT}(Z(d,r))$ via Kraśkiewicz's insertion

Since $w^{(d,r)}$ is vexillary, the unique standard decomposition tableaux $P(w^{(d,r)})$ has shape $Z(d,r)$ and can be nicely described.

The insertion tableau for $w^{(3,2)}$

$$P(w^{(3,2)}) = \begin{array}{|c|c|c|c|c|c|c|}
\hline
4 & 3 & 0 & 1 & 2 & 3 & 4 \\
\hline
\multicolumn{1}{c|}{} & 3 & 0 & 1 & 2 & 3 & \multicolumn{1}{c}{} \\
\cline{2-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 0 & 1 & 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{3-5}
\end{array}$$

Corollary

By restricting to the recording tableaux $Q(\mathbf{a})$*, Kraśkiewicz's insertion gives a bijection* $\mathbf{a} \mapsto Q(\mathbf{a})$ *between* $\mathrm{Red}(w^{(d,r)})$ *and* $\mathrm{SYT}(Z(d,r))$*.*

Proof sketch

We have now finished the second row of

$$\begin{tikzcd}
\mathrm{SYT}(\lambda) \arrow[r, leftrightarrow] & \mathrm{SYT}(Z(d,r))|_\lambda \arrow[r, leftrightarrow] \arrow[d, hook, "\subseteq"] & \mathrm{Red}(w^\lambda) \arrow[r, leftrightarrow] \arrow[d, hook, "\subseteq"] & \mathrm{BS}(Z(d,r))|_\lambda \arrow[r, leftrightarrow] \arrow[d, hook, "\subseteq"] & \mathrm{BS}(\lambda) \\
 & \mathrm{SYT}(Z(d,r)) \arrow[r, leftrightarrow] & \mathrm{Red}(w^{(d,r)}) \arrow[r, leftrightarrow] & \mathrm{BS}(Z(d,r)) &
\end{tikzcd}$$

For an arbitrary λ, choose r large enough such that $\lambda \subset Z(d,r)$.
The choice of r will not matter for the bijection.
We now describe $\mathrm{SYT}(\lambda) \to \mathrm{SYT}(Z(d,r))|_\lambda$ and $\mathrm{BS}(\lambda) \to \mathrm{BS}(Z(d,r))|_\lambda$.

$\mathsf{SYT}(\lambda) \to \mathsf{SYT}(Z(d,r))|_\lambda$

For $T \in \mathsf{SYT}(\lambda)$, we pad it to obtain $T^+ \in \mathsf{SYT}(Z(d,r))$. We pad it from left to right, from top to bottom.

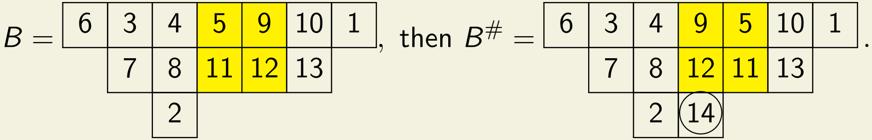
Example: padding a standard shifted tableau

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 9 \\ \hline \multicolumn{1}{c|}{} & 4 & 7 & \multicolumn{3}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 8 & \multicolumn{3}{c}{} \\ \cline{3-3} \end{array}, \text{ then } T^+ = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 9 & \textcircled{10} \\ \hline \multicolumn{1}{c|}{} & 4 & 7 & \textcircled{11} & \textcircled{12} & \textcircled{13} & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & 8 & \textcircled{14} & \textcircled{15} & \multicolumn{2}{c}{} \\ \cline{3-5} \end{array}.$$

$\mathrm{BS}(\lambda) \to \mathrm{BS}(Z(d,r))|_\lambda$

There exists a way to pad balanced shifted tableaux as well. For $B \in \mathrm{BS}(\lambda)$, we pad it from left to right, from top to bottom. At each step, when we add a box to column j, interchange columns j and $j+1$.

Example: padding a balanced shifted tableau

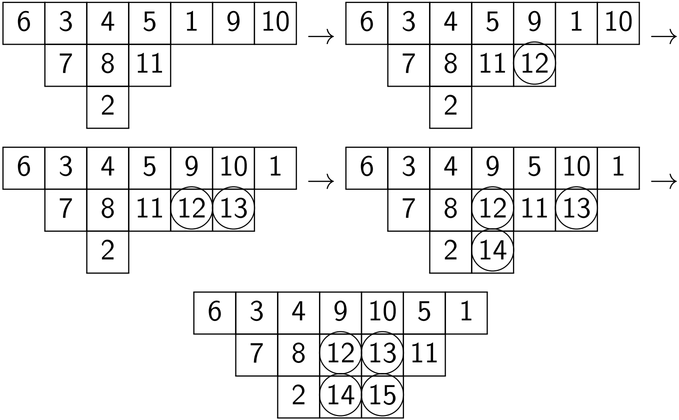


$$B = \begin{array}{ccccccc} 6 & 3 & 4 & \colorbox{yellow}{5} & \colorbox{yellow}{9} & 10 & 1 \\ & 7 & 8 & \colorbox{yellow}{11} & \colorbox{yellow}{12} & 13 & \\ & & 2 & & & & \end{array}, \text{ then } B^{\#} = \begin{array}{ccccccc} 6 & 3 & 4 & \colorbox{yellow}{9} & \colorbox{yellow}{5} & 10 & 1 \\ & 7 & 8 & \colorbox{yellow}{12} & \colorbox{yellow}{11} & 13 & \\ & & 2 & \textcircled{14} & & & \end{array}.$$

Lemma (G-Gao-Gao 2022)

$B^{\#}$ will remain balanced under this process.

An example: $\mathrm{BS}(\lambda) \to \mathrm{BS}(Z(d, r))|_\lambda$



An example

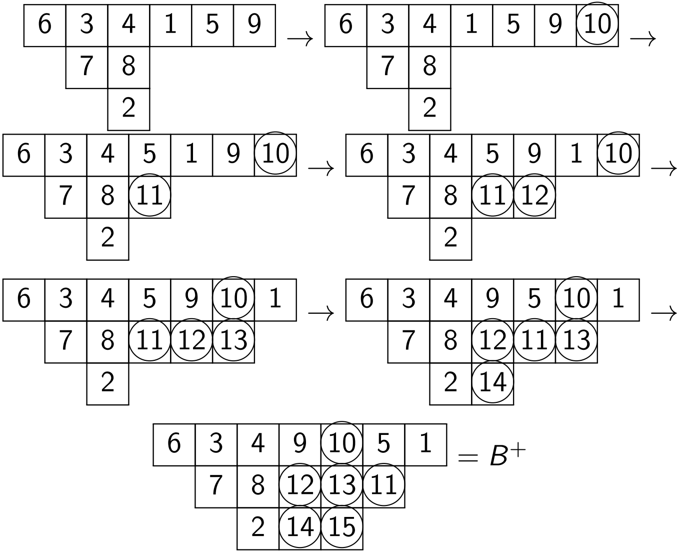
Let's start with a balanced shifted tableau

$$B = \begin{array}{cccccc} \boxed{6} & \boxed{3} & \boxed{4} & \boxed{1} & \boxed{5} & \boxed{9} \\ & \boxed{7} & \boxed{8} & & & \\ & & \boxed{2} & & & \end{array}.$$

We have $\lambda = (6, 2, 1)$ and choose $Z(3, 2)$.

An example: $\mathrm{BS}(\lambda) \to \mathrm{BS}(Z(d,r))|_\lambda$

We pad it to $B^+ \in \mathrm{BS}(Z(d,r))|_\lambda$:



An example: $\mathrm{BS}(Z(d,r))|_\lambda \to \mathrm{Red}(w^{(d,r)})$

$$B^+ = \begin{array}{|c|c|c|c|c|c|c|} \hline 6 & 3 & 4 & 9 & \textcircled{10} & 5 & 1 \\ \hline \multicolumn{1}{c|}{} & 7 & 8 & \textcircled{12} & \textcircled{13} & \textcircled{11} & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & \textcircled{14} & \textcircled{15} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-5} \end{array}$$

gives a reflection order

$$12345 \xrightarrow{e_3-e_2} 13245 \xrightarrow{e_1} \bar{1}3245 \xrightarrow{e_3+e_1} 3\bar{1}245 \xrightarrow{e_3} \bar{3}\bar{1}245 \xrightarrow{e_3-e_1} \bar{1}\bar{3}245$$

$$\xrightarrow{e_3+e_2} \bar{1}2\bar{3}45 \xrightarrow{e_2+e_1} 2\bar{1}\bar{3}45 \xrightarrow{e_2} \bar{2}\bar{1}\bar{3}45 \xrightarrow{e_4+e_3} \bar{2}\bar{1}4\bar{3}5 \xrightarrow{e_5+e_3} \bar{2}\bar{1}45\bar{3}$$

$$\xrightarrow{e_2-e_1} \bar{1}\bar{2}45\bar{3} \xrightarrow{e_4+e_2} \bar{1}4\bar{2}5\bar{3} \xrightarrow{e_5+e_2} \bar{1}45\bar{2}\bar{3} \xrightarrow{e_4+e_1} 4\bar{1}5\bar{2}\bar{3} \xrightarrow{e_5+e_1} 45\bar{1}\bar{2}\bar{3},$$

for which we read off

$$\mathbf{a} = 201012103412312 \in \mathrm{Red}(w^{(3,2)}).$$

An example: $\mathrm{Red}(w^{(d,r)}) \to \mathrm{SYT}(Z(d,r))|_\lambda$

Kraśkiewicz's insertion of $\mathbf{a} = 201012103412312$ gives

$$P^{(0)} = \emptyset \qquad\qquad Q^{(0)} = \emptyset$$

$$P^{(3)} = \begin{array}{|c|c|c|}\hline 2 & 0 & 1 \\ \hline \end{array} \qquad\qquad Q^{(3)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$$

$$P^{(4)} = \begin{array}{|c|c|c|}\hline 2 & 1 & 0 \\ \hline \multicolumn{1}{c|}{} & 0 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \qquad\qquad Q^{(4)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}$$

$$P^{(5)} = \begin{array}{|c|c|c|c|}\hline 2 & 1 & 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 0 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \qquad\qquad Q^{(5)} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$P^{(15)} = \begin{array}{|c|c|c|c|c|c|c|}\hline 4 & 3 & 0 & 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & 0 & 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 0 & 1 & 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-5} \end{array} \qquad Q^{(15)} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 6 & 9 & 10 \\ \hline \multicolumn{1}{c|}{} & 4 & 7 & 11 & 12 & 13 & \multicolumn{1}{c}{} \\ \cline{2-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & 14 & 15 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-5} \end{array}$$

An example: $\mathrm{SYT}(Z(d,r))|_\lambda \to \mathrm{SYT}(\lambda)$

Delete the largest entries from $Q(\mathbf{a})$ until $|\lambda|$ to get

$$T^+ = \begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 5 & 6 & 9 & 10 \\
\hline
\multicolumn{1}{c|}{} & 4 & 7 & 11 & 12 & 13 & \multicolumn{1}{|c}{} \\
\cline{2-6}
\multicolumn{2}{c|}{} & 8 & 14 & 15 & \multicolumn{2}{|c}{} \\
\cline{3-5}
\end{array}, \quad T = \begin{array}{|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 5 & 6 & 9 \\
\hline
\multicolumn{1}{c|}{} & 4 & 7 & \multicolumn{3}{|c}{} \\
\cline{2-3}
\multicolumn{2}{c|}{} & 8 & \multicolumn{3}{|c}{} \\
\cline{3-3}
\end{array}.$$

We now completed the bijection

Example: the bijection $\mathrm{BS}(\lambda) \to \mathrm{SYT}(\lambda)$

$$B = \begin{array}{|c|c|c|c|c|c|}
\hline
6 & 3 & 4 & 1 & 5 & 9 \\
\hline
\multicolumn{1}{c|}{} & 7 & 8 & \multicolumn{3}{|c}{} \\
\cline{2-3}
\multicolumn{2}{c|}{} & 2 & \multicolumn{3}{|c}{} \\
\cline{3-3}
\end{array} \quad T = \begin{array}{|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 5 & 6 & 9 \\
\hline
\multicolumn{1}{c|}{} & 4 & 7 & \multicolumn{3}{|c}{} \\
\cline{2-3}
\multicolumn{2}{c|}{} & 8 & \multicolumn{3}{|c}{} \\
\cline{3-3}
\end{array}$$

Some remarks

There are other notions of "balanced tableaux" in the literature, including

- "balanced labeling" by Fomin-Greene-Reiner-Shimozono,
- and its type B analogue by Zachary Hamaker.

However,

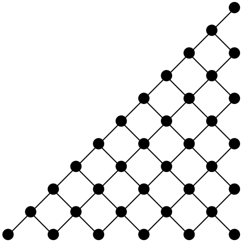
- The above definitions are defined on Rothe diagrams.
- Our definition focuses on tableau and the hook length formula.

Some remarks

What about other root systems?

- Balanced tableaux $\Longleftrightarrow$ reflection order;
- Standard tableaux $\Longleftrightarrow$ linear extension of root poset.

Example: type B_n root poset ($\cong Z(n,0)$)



Conjecture: # reflection orders = # linear extension of root poset?

Some remarks

Unfortunately, $\#\,\mathrm{Red}(w_0(D_4)) = 2316$ and the number of linear extensions of the root poset is $e(\Phi(D_4)^+) = 2400$.

These two quantities also fail to be equal in F_4.

Conjecture (Stanley 1984)

For any Coxeter group W and $J \subset S$,

$$\#\,\mathrm{Red}(w_0^J) \leq e(\Phi_J^+).$$

Thanks!

Thank you for listening!