Sandpile group of Cayley graph of $\mathbb{F}_2^r$

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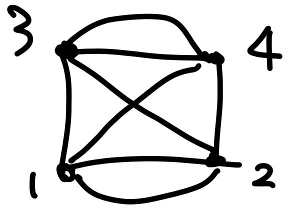
1. Sandpile group of graph G.

- Graph $G=(V,E)$ connected, undirected, no self loop,

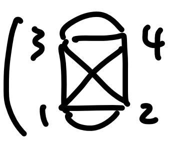
  parallel edges  OK

Ex: 

- Laplacian matrix $L(G)$ $n\times n$ integer matrix

  $(n = \# V)$

$$L(G)_{i,j} = \begin{cases} \deg_G(i) & \text{if } i=j \\ -(\#\text{ edges } i - j) & \text{if } i\neq j \end{cases}$$

Ex: $L\Big($  $\Big) =$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{pmatrix} 4 & -2 & -1 & -1 \\ -2 & 4 & -1 & -1 \\ -1 & -1 & 4 & -2 \\ -1 & -1 & -2 & 4 \end{pmatrix} \rightarrow \text{singular matrix}$$

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \ker L(G)$$

- Sandpile group.

Consider $\mathbb{Z}^n \xrightarrow{L(G)} \mathbb{Z}^n$

We define the sandpile group $K(G)$ as the torsion part of coker $L(G)$

$\ker L(G) = \mathbb{Z}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Fact: If G is connected rank $L(G) = n-1$.

$\operatorname{im} L(G) \cong \mathbb{Z}^{n-1} \subset \mathbb{Z}^n$

lattice

$\operatorname{coker} L(G) = \mathbb{Z}^n / \operatorname{im} L(G) \cong \mathbb{Z} \oplus K(G)$

finite ab. gp.

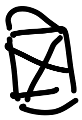
- Equivalently, one can compute $K(G)$ by putting $L(G)$ into a Smith normal form (SNF)

$$P \cdot L(G) \cdot Q = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_{n-1} & \\ & & & & 0 \end{bmatrix}$$ where $d_i \mid d_{i+1}$

$P, Q \in SL_n(\mathbb{Z})$

unique.

Then $K(G) \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}/d_i\mathbb{Z}$

<u>Ex</u>: $G =$  $\quad L(G) \xrightarrow{\text{SNF}} \begin{bmatrix} 1 & & & \\ & 3 & & \\ & & 12 & \\ & & & 0 \end{bmatrix}$

$$K(G) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$$

$$\cong (\mathbb{Z}/3\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z}$$

- $K(G)$ is an important graph invariant

For example, $\# K(G) = \#$ spanning trees in G
[Khirkoff's matrix tree Thm]

We want to know not only the size, but the abelian gp. structure of $K(G)$

$$K(G) = \bigoplus_{\text{prime } p} \underbrace{\bigoplus_{e \geq 1} (\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)}}_{\mathrm{Syl}_p(K(G))}$$

$K(G)$ is only known for certain special graphs

e.g. complete graph K_n, Cycle, wheel graphs etc.

Not even known for hypercubes.

↳ part of talk today!

2. Cayley graph of $\mathbb{F}_2^n$

(generalization of hypercube)

Def: Given a finite group Γ and a multi-set

$M = \{v_1, v_2, \dots, v_n\} \in \Gamma$ that is closed under

inversion, i.e. $v \in M \iff v^{-1} \in M$

(with same multiplicity)

The Cayley graph $G(\Gamma, M)$ has

- vertices $V = \Gamma$
- edges $E = \{ g \xrightarrow{v_i} v_i g \}_{\substack{i=1,2,\dots n \\ g \in \Gamma}}$

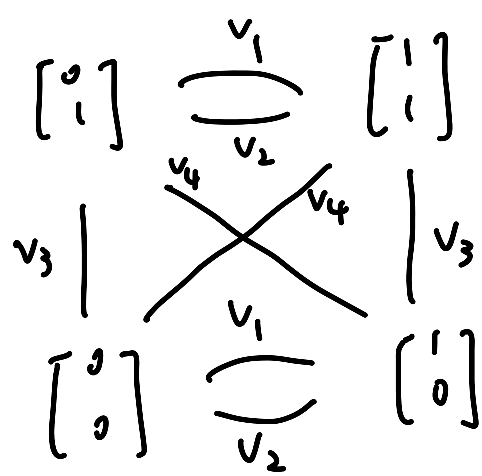
<u>Note:</u> If $g \xrightarrow{v_i} v_i g$, then $v_i g \xrightarrow{v_i^{-1}} g$

View them as one edge

$G(\Gamma, M)$ is undirected

<u>Ex1</u> $\Gamma = \mathbb{F}_2^2$, $M = \begin{array}{c} \left[\begin{array}{c|c|c|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \\ \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \end{array}$

$G = G(\Gamma, M)$



<u>Ex2:</u> Hypercube graph $Q_n = G(\mathbb{F}_2^n, I_n)$ is an example of a Cayley graph of $\mathbb{F}_2^n$

What do we know about $K(G(\mathbb{F}_2^r, M))$?

<u>Thm</u>: [Bai '02] If we know the eigenvalues of $L(G(\mathbb{F}_2^r, M))$ given by $(0=)\lambda_1 \le \lambda_2 \le \dots \le \lambda_{2^r}$

Then $$\mathrm{Syl}_p(K(G(\mathbb{F}_2^r, M))) = \mathrm{Syl}_p\Big(\bigoplus_{i\ge 2} \mathbb{Z}/\lambda_i\mathbb{Z}\Big)$$

for any prime $p \ne 2$.

<u>Ex</u>: $G=$  $\quad K(G) \cong \underbrace{(\mathbb{Z}/3\mathbb{Z})^2}_{\mathrm{Syl}_3} \oplus \underbrace{\mathbb{Z}/4\mathbb{Z}}_{\mathrm{Syl}_2}$

$\lambda = 0, 4, 6, 6 \qquad \bigoplus_{\lambda} \mathbb{Z}/\lambda\mathbb{Z} \cong \underbrace{(\mathbb{Z}/3\mathbb{Z})^2}_{\mathrm{Syl}_3} \oplus \underbrace{(\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z}}_{\mathrm{Syl}_2}$

In fact, the eigenvalues/eigenbasis of $L(G(\mathbb{F}_2^r, M))$ are well-understood.

They are labelled by elements $u \in \mathbb{F}_2^r$

eigenbasis $f_u = \sum_{v \in \mathbb{F}_2^r} (-1)^{u \cdot v} e_v$

$$\lambda_{u,M} = n - \sum_{v_i \in M} (-1)^{u \cdot v_i}$$

So we know $Syl_p(K(G))$ for all $p \neq 2$

But $Syl_2(K(G))$ is wide open.

$$\parallel$$

$$\bigoplus_{e \geq 1} (\mathbb{Z}/2^e\mathbb{Z})^{m(e)}$$

3. Main results

We are interested in two statistics

① The number of Sylow-2 components

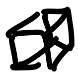
$$d(M) = \sum_{e \geq 1} m(e)$$

powers of 2

② The top sylow-2 factor $V_2(C_{max})$

where C_{max} is the maximum cyclic factor of $K(G(\mathbb{F}_2^n, M))$

Ex:

	G	$Syl_2(K(G))$	$d(M)$	$v_2(C_{max})$
$r=2$	$\square$	$\mathbb{Z}_4$	1	2
$r=3$		$\mathbb{Z}_2\ \mathbb{Z}_8^2$	3	3
$r=4$ gen.	Q_4	$\mathbb{Z}_2^2\ \mathbb{Z}_8^4\ \mathbb{Z}_{32}$	7	5
$r=2$		$\mathbb{Z}_4$	1	2
$r=3$ not generic	K_8	$\mathbb{Z}_8^6$	6	3

① **Thm 1**: $d(M) = 2^{r-1}-1$ if $\sum_{v\in M} v \neq 0$. $\Rightarrow$ M is "generic"

Thm 2: If M is not generic, i.e. $\sum_{v \in M} v = 0$

Then $d(M) \geq 2^{r-1}$

② **Thm 3**: C_{MAX} can be determined by the following process:

(1) Draw a hypercube Q_r.

Write $\frac{1}{\lambda_{u,M}}$ on the vertices of Q_r

(except $u = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ since $\lambda_{\vec{0},M} = 0$)

(2) Record the average of $\frac{1}{\lambda_{u,M}}$'s on

all faces of Q_r with codim ≥ 2

that pass through $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

(3) ———————— 〃 ————————

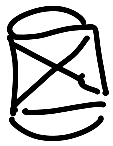
all facets (codim $=1$) of Q_r

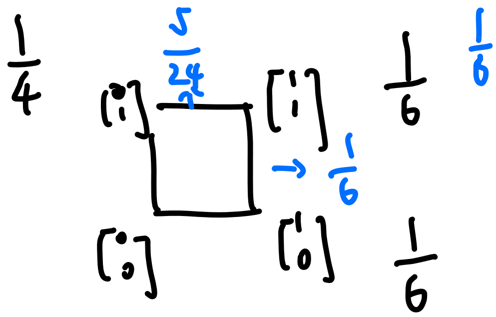
———— 〃 ———— $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, then multiply by 2

(4) C_{MAX} is the common denominator

of all the numbers above

Ex:

G = 



$$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\boxed{K(G) = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}}$$

C_{MAX} is common denominator of $\frac{1}{6}, \frac{5}{12}, \frac{1}{3}$

$\Rightarrow C_{MAX} = 12.$

Note. This is compatible with Thm 0 about Sylp. (why?)

Thm 4 When $G = Q_n$ hypercube, the number C_{MAX} is calculatable.

$$v_2(C_{MAX}) = \max_{1 \le a \le n-1} \{a + v_2(a),\ n + v_2(n) - 1\}$$

4. Key techniques

① Induction on Quotient of Cayley Graphs [Iga, Yuen et. al. '22+]

(For proof of Thm 2)

Def: Given Cayley graph $G(\Gamma, M)$ and a normal subgp. $U \subset \Gamma$ s.t. $U \cap M = \phi$, the quotient Cayley graph $G/U := G(\Gamma/U, M)$ has

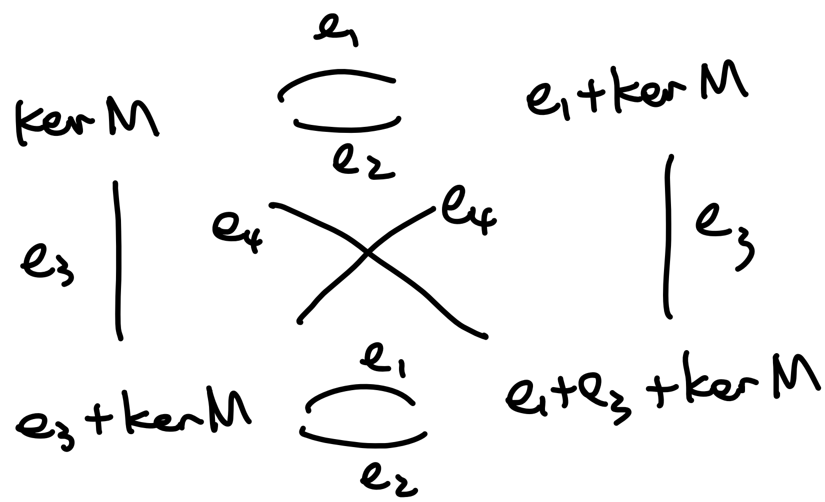
- vertices = {cosets of U}
- edges = $\{ gU \to v_i gU \}_{\substack{v_i \in M \\ gU \text{ coset}}}$

Thm: Any Cayley graph $G(\mathbb{F}_2^n, M)$ is a quotient of the hypercube Q_n

$$G(\mathbb{F}_2^n, M) \cong Q_n / \ker M$$

Ex: $M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$$\ker M = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$



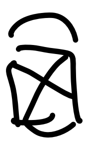
Idea: Induction

$$G \longrightarrow \text{Quotient of } G.$$

② Ring structure on $\mathbb{Z} \oplus K(G)$

[Anzis-Prasad '16] (For proof of Thm 1.3.4)

Although $\mathbb{Z} \oplus K(G) \cong \operatorname{coker} L(G)$ is a group, there is a canonical ring structure on it if G is a Cayley Graph of $\mathbb{F}_2^r$

<u>Ex.</u> $G =$  $M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $\quad \mathbb{Z}^4 \xrightarrow{L(G)} \mathbb{Z}^4$

We can model $\mathbb{Z}^4$ as $\mathbb{Z}[x_1, x_2]/(x_1^2 - 1, x_2^2 - 1)$

$\downarrow$

$\mathbb{Z}$-basis $\quad 1,\ x_1,\ x_2,\ x_1 x_2$

Multiplication by $L(G)$ = Multiplication by

$$4 - 2x_1 - x_2 - x_1 x_2$$

$\Rightarrow \mathbb{Z} \oplus K(G) \cong \mathbb{Z}[x_1, x_2] / (x_1^2-1, x_2^2-1, 4-2x_1-x_2-x_1x_2)$

$\Rightarrow$ Help us apply techniques such as finding a Gröbner basis

③ S_n – Symmetricity on $\mathbb{Z} \oplus K(Q_n)$

(For proof of Thm 4)

$$\mathbb{Z} \oplus K(Q_n) \cong \mathbb{Z}[x_1, \cdots, x_n] / (x_1^2-1, \cdots, x_n^2-1, x_1+\cdots+x_n-n)$$

$\Downarrow$

stable under permutation of variables $x_1 \sim x_n$

5. Past literature and future directions

Past literature :

(1) # of $\mathbb{Z}/2\mathbb{Z}$ components in $\mathrm{Syl}_2(K(Q_n))$

[Bai '02]

(2) Smith normal form of $A(Q_n)$

adj. matrix

(sometimes called the smith gp of Q_n)

[Chandler - Sin - Xiang '15]

Future direction:

(1) More info of $\mathrm{Syl}_2(K(G(\mathbb{F}_2^r, M)))$

or $\mathrm{Syl}_2(K(Q_n))$

(2) Generalize to Cayley graphs of other abelian groups Γ, e.g. $\mathbb{F}_3^r$