Virtual Complete Intersections of Points in $\mathbb{P}^1 \times \mathbb{P}^1$

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   - Virtual Resolutions
   - Virtual Complete Intersections (VCIs)

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   - VCI Existence Cases
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The Biprojective Space $\mathbb{P}^1 \times \mathbb{P}^1$

Definition

The biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ is the set of equivalence classes:

$$\mathbb{P}^1 \times \mathbb{P}^1 := \{((a_0, a_1), (b_0, b_1)) \in \mathbb{C}^2 \times \mathbb{C}^2 \big|_{\substack{(a_0,a_1)\neq(0,0) \\ \text{and } (b_0,b_1)\neq(0,0)}}\}/\sim$$

$x \sim y \iff x = \lambda y$, where $x, y \in \mathbb{P}^1, \lambda \in \mathbb{C}^*$

- $\mathbb{Z}^2$-graded Cox ring $S = \mathbb{C}[x_0, x_1, y_0, y_1]$
- $\deg(x_i) = (1, 0), \deg(y_i) = (0, 1)$
  ex. $x_0^2 y_0 + x_0 x_1 y_1$ has degree $(2, 1)$.

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- $\deg(x_i) = (1, 0), \deg(y_i) = (0, 1)$
  ex. $x_0^2 y_0 + x_0 x_1 y_1$ has degree $(2, 1)$.
- Irrelevant ideal: $B = \langle x_0, x_1\rangle \cap \langle y_0, y_1\rangle \leftrightarrow V(B) = \emptyset$

Virtual Resolutions

While free resolutions encode geometry in projective spaces...

Definition (Berkesch-Erman-Smith, 2017)

A virtual resolution for an ideal I in the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ is a free complex:

$$0 \longleftarrow I \xleftarrow{\varphi_0} S \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \xleftarrow{\varphi_3} \cdots$$

such that

- F_i are free modules for $i \geq 0$
- $\operatorname{ann}\left(\frac{\ker(\varphi_i)}{\operatorname{im}(\varphi_{i+1})}\right) \supseteq B^l$
- $\operatorname{im}(\varphi_1) : B^\infty = I : B^\infty$, where $I : J = \{s \in S | sJ \subseteq I\}$

Virtual Complete Intersection

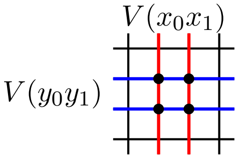
Definition

The variety of an ideal I of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is a **virtual complete intersection (VCI)** if I has a short virtual resolution that is a Koszul complex:

$$S^1 \leftarrow S^2 \leftarrow S^1 \leftarrow 0$$

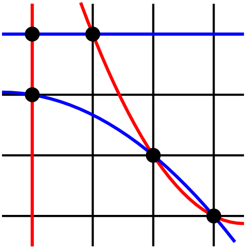
In particular, $V(I) = V(f) \cap V(g)$.

Idea: I has two generators.



$$I(X) = \langle x_0x_1, y_0y_1 \rangle$$

VCI Example



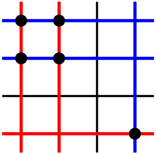
Minimal Virtual Resolution: $S^1 \leftarrow S^2 \leftarrow S^1 \leftarrow 0$

Minimal Free Resolution: $S^1 \leftarrow S^6 \leftarrow S^8 \leftarrow S^3 \leftarrow 0$

Generalized Bézout's Theorem

Theorem

Let $f, g \in k[x_0, x_1, y_0, y_1]$ be bihomogeneous forms. If f and g have multidegree (a, b) and (c, d), then $|V(f) \cap V(g)| = ad + bc$ counting multiplicities.



Red: $x_0x_1(y_0 - y_1)$: $(2, 1)$
Blue:$(x_0 - x_1)y_0y_1$: $(1, 2)$
$1 \cdot 1 + 2 \cdot 2 = 5$ points.

Our Main Results

Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

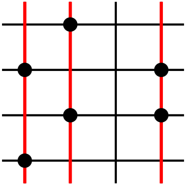
- Existence Case
- Non existence case
- Further conditions on VCIs.

VCI Existence Cases

Theorem

If X has the same number (n) of points in each nonempty vertical (or each horizontal) ruling, it is a VCI.

- Idea: Use Lagrangian interpolation



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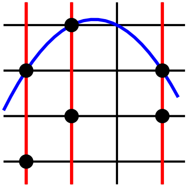

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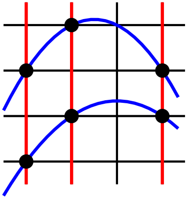
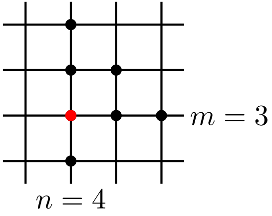


VCI Non-existence Cases

Key invariants

- m: max. number of points on a horizontal ruling
- n: max. number of points on a vertical ruling



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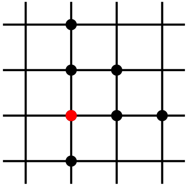

Theorem

If $|X| < mn$, and there is at least one point in X that is on a horizontal ruling with m points and a vertical ruling with n points, then X is not a VCI.

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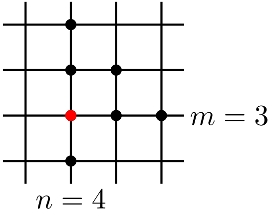

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Proof idea: Bounding with Bézout's Theorem

Conditions on VCIs

Setup: f: (a,b)-form, g: (c,d)-form, $X = V(F) \cap V(g)$.
$\leq m$ points collinear horizontally, $\leq n$ vertically

Theorem

Let X be a VCI with $|X| < mn$.

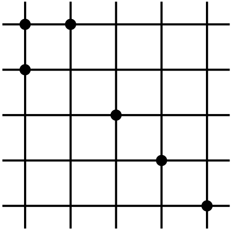
- *f has degree (m,n) and g has vertical and horizontal components exactly on rulings with m and n points*
- $\gcd(m,n)$ *divides* $|X|$
- *If* $\gcd(m,n) = 1$*: g has degree:*

$$(n^{-1}|X| \mod m, \quad m^{-1}|X| \mod n)$$

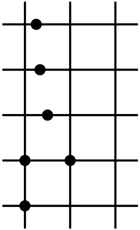
When values of coordinates matter...

Remark

Configuration does not always determine whether a set of points is a VCI. For instance,



In general, not a VCI.



Red:$(2, 1)$; Blue:$(2, 2)$.

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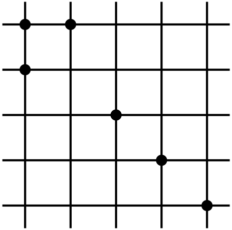

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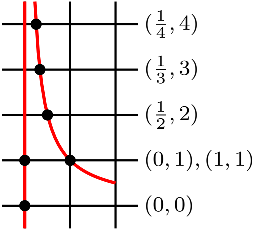

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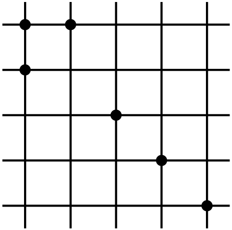

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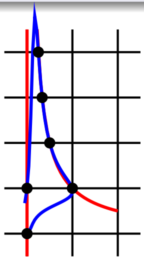

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Acknowledgements

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